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Estimation Methods for Three-Way Repeated Measurements Model

Fatima H. Al-Kanan

Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, Iraq.

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A B S T R A C T

In this paper, we study two estimation methods for a three-way repeated measurements model. We estimated the variance components by using the maximum likelihood method and penalized likelihood method. We determine the moments of each estimator, also we define the likelihood ratio test statistic. We look at the various factors that contribute to accuracy, such as sample size, and biased estimation.

1. Introduction

Repeated measurements analysis is widely used in many fields, for example, in health and life science, psychological, and so on [1]. Repeated measurements is a terminology used to describe data in which the response variable for each experimental units is observed on multiple times and possible under different experimental conditions [2]. We use the mean square error to compare the maximum likelihood estimator with the penalized likelihood estimators[3]. Multi-sample repeated measures ANOVA model is supposed to be a model of repeated measurements a p time points that are obtained from q groups of subjects[4]. Al-mouel and jassim [5] study the two-way multivariate repeated measurements analysis of variance. Al-mouel and jassim [6] study the sphericity test for repeated measures model. Al-mouel and Fakhir [7] study of two-way multivariate repeated measurements analysis of covariance model. Al-mouel and abbas [8] study the one-way multivariate repeated measurements analysis of covariance model. Al-mouel and mustafa [9] study the roy's union-intersection test in one-way multivariate repeated measurements analysis of variance model. Al-mouel

*Corresponding author email : fatima.hameed@uobasrah.edu.iq

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and mustafa [10] study the sphericity test for one-way multivariate repeated measurements analysis of variance model. Al-mouel and Al-Shmailawi [11] study estimation of variance components in the one-way repeated measurements model. Kareem, Jassim, and Al-Hareeb [12] introduced mathematical technique that concerns the finding of minimum or maximum of functions in specific possible area. Jassim and al-mouel [13] propose the lasso method for choice of penalty level and investigate the error of the lasso estimator in repeated measurements model. Jassim and al-mouel [14] propose high-dimensional repeated measurements model. In this paper, we study two estimation methods for three-way repeated measurements model. We look at the various factors that contribute to accuracy, such as sample size, and biased estimation.

2. Setting Up The Model

$\varphi_{ijk} = \mu + \alpha_{i(j)} + \beta_{j(k)} + \lambda_{i(k)} + \epsilon_{ijk}$,

where φ_{ijk} is the response variable, $\alpha_{i(j)}, \beta_{j(k)}$, and $\lambda_{i(k)}$ are the random effects for all $i = 1, \dots, a$, $j = 1, \dots, b$, and $k = 1, \dots, n$ and ϵ_{ijk} is the random error for all $i = 1, \dots, a$, $j = 1, \dots, b$, and $k = 1, \dots, n$.

We assume that $\alpha_{i(j)}$ i.i.d $\sim N(0, \sigma^2_\alpha)$, $\beta_{j(k)}$ i.i.d $\sim N(0, \sigma^2_\beta)$, $\lambda_{i(k)}$ i.i.d $\sim N(0, \sigma^2_\lambda)$, and ϵ_{ijk} i.i.d $\sim N(0, \sigma^2_\epsilon)$.

Now, we start with this identity to partition SS_{total}

$$\varphi_{ijk} - \overline{\varphi}_{...} = (\overline{\varphi}_{ij.} - \overline{\varphi}_{i..}) + (\overline{\varphi}_{.jk} - \overline{\varphi}_{.j.}) + (\overline{\varphi}_{i.k} - \overline{\varphi}_{..k}) \\ + (\varphi_{ijk} - \overline{\varphi}_{ij.} - \overline{\varphi}_{.jk} - \overline{\varphi}_{i.k} + \overline{\varphi}_{i..} + \overline{\varphi}_{.j.} + \overline{\varphi}_{..k} - \overline{\varphi}_{...}),$$

we found that $SS_\alpha, SS_\beta, SS_\lambda$, and SS_ϵ are defined as follows

$SS_\alpha = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (\overline{\varphi}_{ij.} - \overline{\varphi}_{i..})^2$, $SS_\beta = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (\overline{\varphi}_{.jk} - \overline{\varphi}_{.j.})^2$,

$SS_\lambda = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (\overline{\varphi}_{i.k} - \overline{\varphi}_{..k})^2$,

and $SS_\epsilon = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (\varphi_{ijk} - \overline{\varphi}_{ij.} - \overline{\varphi}_{.jk} - \overline{\varphi}_{i.k} + \overline{\varphi}_{i..} + \overline{\varphi}_{.j.} + \overline{\varphi}_{..k} - \overline{\varphi}_{...})^2$.

Table 1. Analysis of variance table

Source of Variance	**D.F.**	**SS**	**MS**	**$F - ratio$**	**F table value**
Random α	$a(b-1)$	SS_α	MS_α	$\frac{MS_\alpha}{MS_\epsilon}$	F[a(b-1),(a-1)(b-1)(n-1);1-α]
Random β	$b(n-1)$	SS_β	MS_β	$\frac{MS_\beta}{MS_\epsilon}$	F[b(n-1),(a-1)(b-1)(n-1);1-α]
Random λ	$n(a-1)$	SS_λ	MS_λ	$\frac{MS_\lambda}{MS_\epsilon}$	F[n(a-1),(a-1)(b-1)(n-1);1-α]

Error	(a-1)(b-1)(n-1)	SS_ϵ	MS_ϵ

3. Maximum Likelihood Estimation

In this section, we estimated the variance components by using the maximum likelihood method.

Theorem 3.1. The variance components estimators are:

$\widehat{\sigma_\epsilon^2} = MS_\epsilon$, $\widehat{n\sigma_\alpha^2 + \sigma_\beta^2} = MS_\alpha - MS_\epsilon$, $\widehat{a\sigma_\beta^2 + \sigma_\lambda^2} = MS_\beta - MS_\epsilon$, and $\widehat{\sigma_\alpha^2 + b\sigma_\lambda^2} = MS_\lambda - MS_\epsilon$.

Proof. The likelihood function is

$$L = \frac{\exp\left[-\frac{1}{2}\left\{\frac{SS_\epsilon}{\sigma_\epsilon^2}+\frac{SS_\alpha}{n\sigma_\alpha^2+\sigma_\beta^2+\sigma_\epsilon^2}+\frac{SS_\beta}{a\sigma_\beta^2+\sigma_\lambda^2+\sigma_\epsilon^2}+\frac{SS_\lambda}{\sigma_\alpha^2+b\sigma_\lambda^2+\sigma_\epsilon^2}+\frac{abn(\bar{\varphi}_{...}-\mu)^2}{n\sigma_\alpha^2+a\sigma_\beta^2+b\sigma_\lambda^2+\sigma_\epsilon^2}\right\}\right]}{(2\pi)^{\frac{1}{2}abn}(\sigma_\epsilon^2)^{\frac{1}{2}v_\epsilon}\left(n\sigma_\alpha^2+\sigma_\beta^2+\sigma_\epsilon^2\right)^{\frac{1}{2}v_\alpha}\left(a\sigma_\beta^2+\sigma_\lambda^2+\sigma_\epsilon^2\right)^{\frac{1}{2}v_\beta}(\sigma_\alpha^2+b\sigma_\lambda^2+\sigma_\epsilon^2)^{\frac{1}{2}v_\lambda}\left(n\sigma_\alpha^2+a\sigma_\beta^2+b\sigma_\lambda^2+\sigma_\epsilon^2\right)^{\frac{1}{2}}},$$

where v_ϵ, v_α, v_β, and v_λ are defined as follows:

$v_\epsilon = (a-1)(b-1)(n-1)$, $v_\alpha = a(b-1)$, $v_\beta = b(n-1)$, and $v_\lambda = n(a-1)$.

The log-likelihood function is

$$\ln(L) = -\frac{1}{2}\Big[abn\ln(2\pi) + v_\epsilon\ln(\sigma_\epsilon^2) + v_\alpha\ln\left(n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2\right) + v_\beta\ln\left(a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2\right) + v_\lambda\ln\left(\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2\right) + \ln\left(n\sigma_\alpha^2 + a\sigma_\beta^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2\right) + \frac{SS_\epsilon}{\sigma_\epsilon^2} + \frac{SS_\alpha}{n\sigma_\alpha^2+\sigma_\beta^2+\sigma_\epsilon^2} + \frac{SS_\beta}{a\sigma_\beta^2+\sigma_\lambda^2+\sigma_\epsilon^2} + \frac{SS_\lambda}{\sigma_\alpha^2+b\sigma_\lambda^2+\sigma_\epsilon^2} + \frac{abn(\bar{\varphi}_{...}-\mu)^2}{n\sigma_\alpha^2+a\sigma_\beta^2+b\sigma_\lambda^2+\sigma_\epsilon^2}\Big],$$

by equating to zero the derivatives $\frac{\partial\ln(L)}{\partial\sigma_\epsilon^2}$, $\frac{\partial\ln(L)}{\partial\left(n\sigma_\alpha^2+\sigma_\beta^2+\sigma_\epsilon^2\right)}$, $\frac{\partial\ln(L)}{\partial\left(a\sigma_\beta^2+\sigma_\lambda^2+\sigma_\epsilon^2\right)}$, and $\frac{\partial\ln(L)}{\partial(\sigma_\alpha^2+b\sigma_\lambda^2+\sigma_\epsilon^2)}$, we have $\widehat{\sigma_\epsilon^2} = MS_\epsilon$, $\widehat{n\sigma_\alpha^2 + \sigma_\beta^2} = MS_\alpha - MS_\epsilon$, $\widehat{a\sigma_\beta^2 + \sigma_\lambda^2} = MS_\beta - MS_\epsilon$, and $\widehat{\sigma_\alpha^2 + b\sigma_\lambda^2} = MS_\lambda - MS_\epsilon$.

Theorem 3.2. The maximum likelihood estimators are unbiased and have these bias equations:

$$E\left(\widehat{\sigma_\epsilon^2}\right) - \sigma_\epsilon^2 = 0,$$
$$E\left(\widehat{n\sigma_\alpha^2 + \sigma_\beta^2}\right) - \left(n\sigma_\alpha^2 + \sigma_\beta^2\right) = 0,$$
$$E\left(\widehat{a\sigma_\beta^2 + \sigma_\lambda^2}\right) - \left(a\sigma_\beta^2 + \sigma_\lambda^2\right) = 0,$$
$$E\left(\widehat{\sigma_\alpha^2 + b\sigma_\lambda^2}\right) - \left(\sigma_\alpha^2 + b\sigma_\lambda^2\right) = 0.$$

Proof.
$$E\left(\widehat{\sigma_\epsilon^2}\right) - \sigma_\epsilon^2 = E(MS_\epsilon) - \sigma_\epsilon^2 = \frac{\sigma_\epsilon^2}{v_\epsilon}v_\epsilon - \sigma_\epsilon^2 = 0,$$

$$E\left(\widehat{n\sigma_\alpha^2 + \sigma_\beta^2}\right) - \left(n\sigma_\alpha^2 + \sigma_\beta^2\right) = E(MS_\alpha - MS_\epsilon) - \left(n\sigma_\alpha^2 + \sigma_\beta^2\right) = \frac{n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2}{v_\alpha}v_\alpha - \frac{\sigma_\epsilon^2}{v_\epsilon}v_\epsilon - \left(n\sigma_\alpha^2 + \sigma_\beta^2\right) = 0,$$

$$E\left(\widehat{a\sigma_\beta^2 + \sigma_\lambda^2}\right) - \left(a\sigma_\beta^2 + \sigma_\lambda^2\right) = E\left(MS_\beta - MS_\epsilon\right) - \left(a\sigma_\beta^2 + \sigma_\lambda^2\right) = \frac{a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2}{v_\beta}v_\beta - \frac{\sigma_\epsilon^2}{v_\epsilon}v_\epsilon - \left(a\sigma_\beta^2 + \sigma_\lambda^2\right) = 0,$$

$$E\left(\widehat{\sigma_\alpha^2 + b\sigma_\lambda^2}\right) - \left(\sigma_\alpha^2 + b\sigma_\lambda^2\right) = E(MS_\lambda - MS_\epsilon) - \left(\sigma_\alpha^2 + b\sigma_\lambda^2\right)$$

$$= \frac{\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2}{v_\lambda} v_\lambda - \frac{\sigma_\epsilon^2}{v_\epsilon} v_\epsilon - \left(\sigma_\alpha^2 + b\sigma_\lambda^2\right)$$
$$= 0.$$

Theorem 3.3. The maximum likelihood estimators have these variances:

$$\operatorname{var}\left(\widehat{\sigma_\epsilon^2}\right) = \frac{2(\sigma_\epsilon^2)^2}{v_\epsilon},$$
$$\operatorname{var}\left(\widehat{n\sigma_\alpha^2 + \sigma_\beta^2}\right) = \frac{2\left(n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2\right)^2}{v_\alpha} + \frac{2(\sigma_\epsilon^2)^2}{v_\epsilon},$$
$$\operatorname{var}\left(\widehat{a\sigma_\beta^2 + \sigma_\lambda^2}\right) = \frac{2\left(a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2\right)^2}{v_\beta} + \frac{2(\sigma_\epsilon^2)^2}{v_\epsilon},$$
$$\operatorname{var}\left(\widehat{\sigma_\alpha^2 + b\sigma_\lambda^2}\right) = \frac{2\left(\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2\right)^2}{v_\lambda} + \frac{2(\sigma_\epsilon^2)^2}{v_\epsilon}.$$

Proof. $\operatorname{var}\left(\widehat{\sigma_\epsilon^2}\right) = \operatorname{var}(MS_\epsilon)$

$$= \left(\frac{\sigma_\epsilon^2}{v_\epsilon}\right)^2 2v_\epsilon$$
$$= \frac{2(\sigma_\epsilon^2)^2}{v_\epsilon},$$

$$\operatorname{var}\left(\widehat{n\sigma_\alpha^2 + \sigma_\beta^2}\right) = \operatorname{var}(MS_\alpha) + \operatorname{var}(MS_\epsilon)$$
$$= \left(\frac{n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2}{v_\alpha}\right)^2 2v_\alpha + \operatorname{var}\left(\widehat{\sigma_\epsilon^2}\right)$$
$$= \frac{2\left(n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2\right)^2}{v_\alpha} + \frac{2(\sigma_\epsilon^2)^2}{v_\epsilon},$$

$$\operatorname{var}\left(\widehat{a\sigma_\beta^2 + \sigma_\lambda^2}\right) = \operatorname{var}\left(MS_\beta\right) + \operatorname{var}(MS_\epsilon)$$
$$= \left(\frac{a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2}{v_\beta}\right)^2 2v_\beta + \operatorname{var}\left(\widehat{\sigma_\epsilon^2}\right)$$
$$= \frac{2\left(a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2\right)^2}{v_\beta} + \frac{2(\sigma_\epsilon^2)^2}{v_\epsilon},$$

$$\operatorname{var}\left(\widehat{\sigma_\alpha^2 + b\sigma_\lambda^2}\right) = \operatorname{var}(MS_\lambda) + \operatorname{var}(MS_\epsilon)$$
$$= \left(\frac{\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2}{v_\lambda}\right)^2 2v_\lambda + \operatorname{var}\left(\widehat{\sigma_\epsilon^2}\right)$$
$$= \frac{2\left(\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2\right)^2}{v_\lambda} + \frac{2(\sigma_\epsilon^2)^2}{v_\epsilon}.$$

4. Maximum Penalized Likelihood Estimation

In this section, we estimated the variance components by using the penalized likelihood method.

Theorem 4.1. The variance components estimators are:

$\widehat{\sigma_\epsilon^2} = \frac{SS_\epsilon}{v_\epsilon+2}$, $\widehat{n\sigma_\alpha^2 + \sigma_\beta^2} = \frac{SS_\alpha}{v_\alpha+2} - \frac{SS_\epsilon}{v_\epsilon+2}$, $\widehat{a\sigma_\beta^2 + \sigma_\lambda^2} = \frac{SS_\beta}{v_\beta+2} - \frac{SS_\epsilon}{v_\epsilon+2}$, and $\widehat{\sigma_\alpha^2 + b\sigma_\lambda^2} = \frac{SS_\lambda}{v_\lambda+2} - \frac{SS_\epsilon}{v_\epsilon+2}$.

Proof. The penalized log-likelihood function is

$\ln\left(L_p\right) = \ln(L) + \ln p\left(\sigma_\epsilon^2, \sigma_\alpha^2, \sigma_\beta^2, \sigma_\lambda^2\right),$

where the prior distribution for $\sigma_\epsilon^2, \sigma_\alpha^2, \sigma_\beta^2$, and σ_λ^2 is

$$p(\sigma_\epsilon^2, \sigma_\alpha^2, \sigma_\beta^2, \sigma_\lambda^2) \alpha \frac{1}{\sigma_\epsilon^2 (n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2)(a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2)(\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2)},$$

$$\begin{aligned}\ln(L_p) = -\frac{1}{2}\Big[& abn\ln(2\pi) + v_\epsilon \ln(\sigma_\epsilon^2) + v_\alpha \ln(n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2) + v_\beta \ln(a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2) \\ & + v_\lambda \ln(\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2) + \ln(n\sigma_\alpha^2 + a\sigma_\beta^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2) + \frac{SS_\epsilon}{\sigma_\epsilon^2} + \frac{SS_\alpha}{n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2} \\ & + \frac{SS_\beta}{a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2} + \frac{SS_\lambda}{\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2} + \frac{abn(\overline{\varphi}_{\ldots} - \mu)^2}{n\sigma_\alpha^2 + a\sigma_\beta^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2}\Big] - \ln(\sigma_\epsilon^2) \\ & - \ln(n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2) - \ln(a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2) - \ln(\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2),\end{aligned}$$

the additive penalty term is the log-likelihood for the Bayesian prior density. By equating to zero the derivatives $\frac{\partial \ln(L_p)}{\partial \sigma_\epsilon^2}$, $\frac{\partial \ln(L_p)}{\partial (n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2)}$, $\frac{\partial \ln(L_p)}{\partial (a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2)}$, and $\frac{\partial \ln(L_p)}{\partial (\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2)}$, we have

$\widehat{\sigma_\epsilon^2} = \frac{SS_\epsilon}{v_\epsilon + 2}$, $\widehat{n\sigma_\alpha^2 + \sigma_\beta^2} = \frac{SS_\alpha}{v_\alpha + 2} - \frac{SS_\epsilon}{v_\epsilon + 2}$, $\widehat{a\sigma_\beta^2 + \sigma_\lambda^2} = \frac{SS_\beta}{v_\beta + 2} - \frac{SS_\epsilon}{v_\epsilon + 2}$, and $\widehat{\sigma_\alpha^2 + b\sigma_\lambda^2} = \frac{SS_\lambda}{v_\lambda + 2} - \frac{SS_\epsilon}{v_\epsilon + 2}$.

Theorem 4.2. The maximum penalized likelihood estimators have these bias equations:

$$E(\widehat{\sigma_\epsilon^2}) - \sigma_\epsilon^2 = \frac{\sigma_\epsilon^2}{v_\epsilon + 2} v_\epsilon - \sigma_\epsilon^2,$$

$$E\left(\widehat{n\sigma_\alpha^2 + \sigma_\beta^2}\right) - (n\sigma_\alpha^2 + \sigma_\beta^2) = \frac{n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2}{v_\alpha + 2} v_\alpha - \frac{\sigma_\epsilon^2}{v_\epsilon + 2} v_\epsilon - (n\sigma_\alpha^2 + \sigma_\beta^2),$$

$$E\left(\widehat{a\sigma_\beta^2 + \sigma_\lambda^2}\right) - (a\sigma_\beta^2 + \sigma_\lambda^2) = \frac{a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2}{v_\beta + 2} v_\beta - \frac{\sigma_\epsilon^2}{v_\epsilon + 2} v_\epsilon - (a\sigma_\beta^2 + \sigma_\lambda^2),$$

$$E\left(\widehat{\sigma_\alpha^2 + b\sigma_\lambda^2}\right) - (\sigma_\alpha^2 + b\sigma_\lambda^2) = \frac{\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2}{v_\lambda + 2} v_\lambda - \frac{\sigma_\epsilon^2}{v_\epsilon + 2} v_\epsilon - (\sigma_\alpha^2 + b\sigma_\lambda^2).$$

Proof. $E(\widehat{\sigma_\epsilon^2}) - \sigma_\epsilon^2 = E\left(\frac{SS_\epsilon}{v_\epsilon + 2}\right) - \sigma_\epsilon^2$

$$= \frac{\sigma_\epsilon^2}{v_\epsilon + 2} v_\epsilon - \sigma_\epsilon^2,$$

$$E\left(\widehat{n\sigma_\alpha^2 + \sigma_\beta^2}\right) - (n\sigma_\alpha^2 + \sigma_\beta^2) = E\left(\frac{SS_\alpha}{v_\alpha + 2} - \frac{SS_\epsilon}{v_\epsilon + 2}\right) - (n\sigma_\alpha^2 + \sigma_\beta^2)$$

$$= \frac{n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2}{v_\alpha + 2} v_\alpha - \frac{\sigma_\epsilon^2}{v_\epsilon + 2} v_\epsilon - (n\sigma_\alpha^2 + \sigma_\beta^2),$$

$$E\left(\widehat{a\sigma_\beta^2 + \sigma_\lambda^2}\right) - (a\sigma_\beta^2 + \sigma_\lambda^2) = E\left(\frac{SS_\beta}{v_\beta + 2} - \frac{SS_\epsilon}{v_\epsilon + 2}\right) - (a\sigma_\beta^2 + \sigma_\lambda^2)$$

$$= \frac{a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2}{v_\beta + 2} v_\beta - \frac{\sigma_\epsilon^2}{v_\epsilon + 2} v_\epsilon - (a\sigma_\beta^2 + \sigma_\lambda^2),$$

$$E\left(\widehat{\sigma_\alpha^2 + b\sigma_\lambda^2}\right) - \left(\sigma_\alpha^2 + b\sigma_\lambda^2\right) = E\left(\frac{SS_\lambda}{v_\lambda + 2} - \frac{SS_\epsilon}{v_\epsilon + 2}\right) - \left(\sigma_\alpha^2 + b\sigma_\lambda^2\right)$$

$$= \frac{\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2}{v_\lambda + 2} v_\lambda - \frac{\sigma_\epsilon^2}{v_\epsilon + 2} v_\epsilon - \left(\sigma_\alpha^2 + b\sigma_\lambda^2\right).$$

Theorem 4.3. The maximum penalized likelihood estimators have these variances:

$$\operatorname{var}\left(\widehat{\sigma_\epsilon^2}\right) = \left(\frac{\sigma_\epsilon^2}{v_\epsilon + 2}\right)^2 2v_\epsilon,$$

$$\operatorname{var}\left(\widehat{n\sigma_\alpha^2 + \sigma_\beta^2}\right) = \frac{2v_\alpha\left(n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2\right)^2}{(v_\alpha + 2)^2} + \left(\frac{\sigma_\epsilon^2}{v_\epsilon + 2}\right)^2 2v_\epsilon,$$

$$\operatorname{var}\left(\widehat{a\sigma_\beta^2 + \sigma_\lambda^2}\right) = \frac{2v_\beta\left(a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2\right)^2}{\left(v_\beta + 2\right)^2} + \left(\frac{\sigma_\epsilon^2}{v_\epsilon + 2}\right)^2 2v_\epsilon,$$

$$\operatorname{var}\left(\widehat{\sigma_\alpha^2 + b\sigma_\lambda^2}\right) = \frac{2v_\lambda\left(\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2\right)^2}{(v_\lambda + 2)^2} + \left(\frac{\sigma_\epsilon^2}{v_\epsilon + 2}\right)^2 2v_\epsilon.$$

Proof. $\operatorname{var}\left(\widehat{\sigma_\epsilon^2}\right) = \operatorname{var}\left(\frac{SS_\epsilon}{v_\epsilon+2}\right)$

$$= \left(\frac{\sigma_\epsilon^2}{v_\epsilon + 2}\right)^2 2v_\epsilon,$$

$$\operatorname{var}\left(\widehat{n\sigma_\alpha^2 + \sigma_\beta^2}\right) = \operatorname{var}\left(\frac{SS_\alpha}{v_\alpha + 2}\right) + \operatorname{var}\left(\frac{SS_\epsilon}{v_\epsilon + 2}\right)$$

$$= \left(\frac{n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2}{v_\alpha + 2}\right)^2 2v_\alpha + \operatorname{var}\left(\widehat{\sigma_\epsilon^2}\right)$$

$$= \frac{2v_\alpha\left(n\sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\epsilon^2\right)^2}{(v_\alpha + 2)^2} + \left(\frac{\sigma_\epsilon^2}{v_\epsilon + 2}\right)^2 2v_\epsilon,$$

$$\operatorname{var}\left(\widehat{a\sigma_\beta^2 + \sigma_\lambda^2}\right) = \operatorname{var}\left(\frac{SS_\beta}{v_\beta + 2}\right) + \operatorname{var}\left(\frac{SS_\epsilon}{v_\epsilon + 2}\right)$$

$$= \left(\frac{a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2}{v_\beta + 2}\right)^2 2v_\beta + \operatorname{var}\left(\widehat{\sigma_\epsilon^2}\right)$$

$$= \frac{2v_\beta\left(a\sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2\right)^2}{\left(v_\beta + 2\right)^2} + \left(\frac{\sigma_\epsilon^2}{v_\epsilon + 2}\right)^2 2v_\epsilon,$$

$\operatorname{var}\left(\widehat{\sigma_\alpha^2 + b\sigma_\lambda^2}\right) = \operatorname{var}\left(\frac{SS_\lambda}{v_\lambda+2}\right) + \operatorname{var}\left(\frac{SS_\epsilon}{v_\epsilon+2}\right)$

$$= \left(\frac{\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2}{v_\lambda + 2}\right)^2 2v_\lambda + \mathrm{var}\left(\widehat{\sigma_\epsilon^2}\right)$$

$$= \frac{2v_\lambda\left(\sigma_\alpha^2 + b\sigma_\lambda^2 + \sigma_\epsilon^2\right)^2}{(v_\lambda + 2)^2} + \left(\frac{\sigma_\epsilon^2}{v_\epsilon + 2}\right)^2 2v_\epsilon.$$

5. Likelihood Ratio Test

We define the null and alternative hypotheses as follows:

$H_0: \mu = \mu_0$ versus $H_1: \mu \neq \mu_0$.

The likelihood ratio test statistic is

$$\tau(\varphi) = \frac{L(\mu_0|\varphi)}{L(\hat{\mu}|\varphi)}$$

$$= \frac{(2\pi\omega)^{-\frac{abn}{2}} \exp\left(\frac{-\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(\varphi_{ijk} - \mu_0\right)^2}{2\omega}\right)}{(2\pi\omega)^{-\frac{abn}{2}} \exp\left(\frac{-\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(\varphi_{ijk} - \overline{\varphi}_{\dots}\right)^2}{2\omega}\right)}$$

$$= \exp\left[\frac{-\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(\varphi_{ijk} - \mu_0\right)^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(\varphi_{ijk} - \overline{\varphi}_{\dots}\right)^2}{2\omega}\right]$$

$$= \exp\left[\frac{-abn(\overline{\varphi}_{\dots} - \mu_0)^2}{2\omega}\right],$$

where $\overline{\varphi}_{\dots}$ is the maximum likelihood estimator for μ, and $\omega = \sigma_\alpha^2 + \sigma_\beta^2 + \sigma_\lambda^2 + \sigma_\epsilon^2$.

We reject H_0 for small values of $\tau(\varphi)$, the rejection region is

$\left\{\varphi: |\overline{\varphi}_{\dots} - \mu_0| \geq \sqrt{\frac{-2\omega(\log C)}{abn}}\right\}$, where $0 < C < 1$, and $0 < \sqrt{\frac{-2\omega(\log C)}{abn}} < \infty$.

6. The Experiment

The data of the experiment was taken from the laboratory of storage technologies, department of horticulture and landscape engineering, university of basra during the growing season 2020-2021 on sider fruits of red malasi variety, to investigate the effect of treatment with chitosan, type of packing and storage period on sidr fruits, cultivar Malasi red, on the percentage of spoilage [15]. The experiment included two levels of chitosan, also there are three levels of packing, and five levels of storage period.

Table 2. The data

chitosan polymer coating	Type of packaging	storage period				
		before storage	w2	w4	w6	w8
0%	p1	0.00	3.33	5.67	12.67	35.33
	p2	0.00	10.67	25.00	38.33	75.00
	p3	0.00	5.67	12.33	24.67	43.33
1%	p1	0.00	0.33	2.00	5.67	15.33
	p2	0.00	5.33	11.67	21.67	46.67
	p3	0.00	2.33	5.33	14.00	36.67
2%	p1	0.00	0.33	1.33	3.00	11.67
	p2	0.00	3.33	5.67	5.67	12.67
	p3	0.00	1.33	3.33	7.33	21.33

Table 3. ANOVA table

Source of Variance	D.F.	SS	MS	F-ratio	F table value
Random α	6	1301.29472	216.882453	2.17791	2.74131
Random β	12	5466.3438	455.52865	4.57437	2.42466
Random λ	10	2772.81513	277.281513	2.78443	2.49351
Error	16	1593.32491	99.582806875		

since the calculated test statistics $F_{\beta} = \frac{MS_{\beta}}{MS_{\epsilon}}$, and $F_{\lambda} = \frac{MS_{\beta}}{MS_{\epsilon}}$ greater than the table values, then the null hypotheses H_0^{β}: all $\mathrm{a}\sigma_{\beta}^2 + \sigma_{\lambda}^2 = 0$, and H_0^{λ}: all $\sigma_{\alpha}^2 + \mathrm{b}\sigma_{\lambda}^2 = 0$, are rejected.

Table 4. Variance components estimators

parameter	maximum likelihood estimator	maximum penalized likelihood estimator
σ_{ϵ}^2	99.582806875	88.5180505
$n\sigma_{\alpha}^2 + \sigma_{\beta}^2$	117.299646125	162.66184
$a\sigma_{\beta}^2 + \sigma_{\lambda}^2$	355.945843125	390.4531285714
$\sigma_{\alpha}^2 + b\sigma_{\lambda}^2$	177.698706125	231.0679275

The results in the above table showed the accuracy of the repeated measurements model because the variance of the error is smaller than other variances in the percentage of spoilage.

7. Conclusion

Different estimation methods have different assumptions and handle data differently, so it's important to choose a method that aligns with the data characteristics. If the focus is on estimating fixed effects and hypothesis testing, MLE or REML may be appropriate. If the interest lies in quantifying uncertainty and incorporating prior knowledge, Bayesian or maximum penalized likelihood estimation may be more suitable. The penalized likelihood estimator offers several advantages over traditional methods for variance component estimation. Firstly, it can effectively handle unbalanced data, this ensures that each observation has a fair influence on the estimation. Additionally, the penalized likelihood estimator can handle small sample sizes.

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طرق التقدير لنموذج القياسات المتكررة ثلاثي الاتجاهات

فاطمة حميد الكنعان

قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة البصرة، البصرة، العراق.

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الملخص

في هذا البحث، قمنا بدراسة طريقتين لتقدير نموذج القياسات المتكررة الثلاثية. وقد قمنا بتقدير مكونات التباين باستخدام طريقة الاحتمال الاعظم وطريقة الاحتمال الجزائي. نقوم بتحديد العزوم لكل مقدر، كما نقوم بتحديد إحصائية اختبار النسبة الاحتمالي. نحن نبحث العوامل المختلفة التي تساهم في الدقة، مثل حجم العينة والتقدير المتحيز.

*Corresponding author email : fatima.hameed@uobasrah.edu.iq

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