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Spectral Petrov-Galerkin Method for Solving the Two-Dimensional Integral Equations of the Second Kind

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ABSTRACT

Using a spectral Petrov-Galerkin approach (SPGM), the main objective of this research is to propose a numerical solution for the two-dimensional (2D) linear Volterra and mixed Volterra-Fredholm integral equations (VIEs and MIEs, respectively) type. We solving four examples that illustrate the results of the method compared with other methods in terms of excellent accuracy, powerful method and less error it provides for this type of integral equations (IEs).

1. Introduction

In many scientific domains, including biophysics and biomedicine, IEs are crucial. Numerous varieties of integral equations have been examined to address a range of issues in applied mathematics, including modeling and informatics [10,4].

Two-dimensional 2D linear IEs are the topic of this paper [4]

$$u(x,y) = f(x,y) + \int_0^y \int_0^x k_1(x,y,z,s)u(z,s)dzds, \qquad (1.1a)$$

or

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$$u(x,y) = f(x,y) + \int_0^y \int_\Omega k_2(x,y,z,s)u(z,s)dzds. \quad (1.1b)$$

VIE and MIE respectively. where $(x,y) \in D$ (Ω - a compact subset of Euclidean space $\mathbb{R}^k$ or compact manifold) and f is the given function in domain D and u is unknown function in D . The given kernel k is (real-values) sufficiently smooth defined on $S \coloneqq \{(x,y,z,s): 0 \le z \le x \le a\,, 0 \le s \le y \le b\}$. In many situations, solving IEs analytically can be difficult; Therefore, it is better to find numerical solutions. The numerical solutions of 2D IEs have been the subject of numerous research attempts. He was presented Han Guoqiang in (1994) numerical solution of 2D linear VIE_S by collocation and iterated collocation is considered. The asymptotic error expansion of the iterated collocation was obtained [8]. The Guoqiang et al. numerical solution of 2D nonlinear VIEs by collocation and iterated collocation method was introduced in (2000). Where it was asymptotic error expansion of iterated collocation solution is obtained by using Richardson's extrapolation [9]. Nemati and Ordokhani in (2012) presented an approximate solution for the 2D nonlinear VIE_S using Legendre orthogonal polynomials [18]. Somayyeh et al in (2013) found approximate solution for linear 2D VIEs, based on the interval approximation of the true solution by truncated Chebyshev series [7]. In (2021), Mikkola introduced an iterative numerical technique for approximating solutions of the second kind of 2D MIEs. The procedure made use of a suitable cubic formula and successive Mann-type approximations [15]. 2D MIEs were solved by Mirzaei and Samadyar (2018) using an effective numerical technique. This technique uses the collocation method to operate on 2D orthogonal Bernstein polynomials (2D-OBPs). This easy technique has been used to solve an algebraic system of equations that is an engineering study issue that can be solved with the fewest possible steps [16]. In order to solve 2D linear and nonlinear VIEs, Yubin Pan and Jin Huang Fei (2020) proposed using numerical quadrature. Initially, they extend the one-dimension (1D) quadratic formula to the 2D scenario, along with the associated error asymptotic expansion. They then built an extrapolation method and a histogram based on the quadrature and error expansion formula [19]. AL – Humedi and Munaty in (2022) used the SPGM to solve the first type of VIEs [1]. A numerical method for 2D-VIEs is presented by Hafid-Lib and et (2023). They were able to develop an assembly solution by employing a technique that makes use of Taylor polynomials in order to approximate the 2D-VIEs solution [14]. The analytical solution of 2D linear IEs is the goal of this research. We derive the SPGM based on Chebyshev and Laguerre orthogonal polynomials to approximate the solution of these equations. This method has the ability to produce fewer errors and increase accuracy compared to other analysis methods. The search order is as follows. Also, define the polynomials in the (1D) and (2D) in the two and three sections. The Chebyshev and Laquerre polynomials will be used to generate the SPGM, and will explain how this approach yields an approximative solution in section four. In section five, will present a few instances that illustrate how the new approach performs in comparison to the outcomes of previous approaches. Section six will provide the research and work's conclusions.

2. Orthogonality Polynomial on Intervals [1]

A sequence of polynomials $\{\varphi_n(x)\}_{n=0}^{\infty}$ with a degree $[\varphi_n(x)] = n$ for each n is called orthogonal with respect to the weight function $w(x)$ on the $C[a,b]$ if the inner product of the polynomials φ_i, φ_j

$$\langle \varphi_i,\ \varphi_j \rangle w \;=\; \int_a^b \varphi_i(x)\varphi_j(x)w(x)\,dx \;=\; \delta_{i,j}\,,$$

with $\delta_{i,j} = \begin{cases} 0 & ; i \neq j \\ 1 & ; i = j \end{cases}$

where $\delta_{i,j}$ is the Kronecker delta function, where $i, j \in \mathbb{N}$.

3. Two-Dimensional Orthogonality Polynomials [21]

By considering 1D orthogonality polynomials, we define an $(n+1)^2$ set of 2D orthogonality polynomials as:

$$\varphi_{i,j}(x,y) = \varphi_i(x)\varphi_j(y) \quad , \quad i,j = 0,1,2,\dots,n .$$

Therefore, 2D basis orthogonality vector is as follows:

$$\varphi(x,y) = [\varphi_0(x)\varphi_0(y),\cdots,\varphi_0(x)\varphi_n(y),\ \varphi_1(x)\varphi_0(y),\cdots,\varphi_1(x)\varphi_n(y),\cdots,\ \varphi_n(x)\varphi_0(y),\ \dots,\varphi_n(x)\varphi_n(y)]^t$$

$$= (\varphi_i(x)\otimes\varphi_j(y))^t \quad , \quad i,j = 0,1,2,\dots,n ,$$

in which

$$\varphi_i(x) = [\varphi_0(x),\varphi_1(x),\ \cdots\ ,\varphi_n(x)] \ , \ \ \varphi_j(y) = [\varphi_0(y),\varphi_1(y),\ \cdots\ ,\varphi_n(y)] .$$

are 1D vectors. The weight function in 2D is $w(x,t) = w(x)w(y)$ on the interval $[a,b]\times[a,b]$.

4. The derivation method

By introducing the integral operator k_1 and k_2 defined is

$$K_1u(x,y) = \int_0^y\int_0^x k_1(x,y,z,s)u(z,s)dzds,$$

or

$$K_2u(x,y) = \int_0^y\int_\Omega k_2(x,y,z,s)u(z,s)dzds.$$

Eqs (1.1) can be written in the following forms

$$u(x,y) = f(x,y) + K_1u(x,y), \tag{4.1a}$$

or

$$u(x,y) = f(x,y) + K_2u(x,y), \tag{4.1b}$$

In order to solve Eqs (4.1), we shall be the SPGM.

First let's see how SPGM works numerically, P_{nm} it is a space of polynomials defined in $[0,1]\times[0,1]$ with the maximum degree nm , $\varphi_i(x),\varphi_j(y)$ are Laguerre polynomials. As

$$P_{nm} = span\{\varphi_0(x)\varphi_0(y),\cdots,\varphi_0(x)\varphi_m(y),\varphi_1(x)\varphi_0(y),\cdots,\varphi_1(x)\varphi_m(y),\cdots,\varphi_n(x)\varphi_0(y),\cdots,\varphi_n(x)\varphi_m(y)\}$$

Define the polynomial space as $V_{nm} = \{u(x,y): u\in P_{nm}\}$.

Find $u_{nm}\in V_{nm}$ such that Eqs. (4.1) can be written

$$(u_{nm},v_{nm})_w = (f,v_{nm})_w + (K_1u_{nm},v_{nm})_w \tag{4.2a}$$

or

$$(u_{nm},v_{nm})_w = (f,v_{nm})_w + (K_2u_{nm},v_{nm})_w \tag{4.2b}$$

$\forall\, v_{nm} \in P_{nm}$

where

$$(u_{nm}, v_{nm})_w = \int_a^b \int_c^d u_{nm}(x,y) v_{nm}(x,y) w(x,y) dxdy$$

$$(K_1 u_{nm}, v_{nm})_w = \int_a^b \int_c^d \int_0^y \int_0^x k_1(x,y,z,s) u_{nm}(z,s)\, v_{nm}(x,y) w(x,y) dzdsdxdy$$

or

$$(K_2 u_{nm}, v_{nm})_w = \int_a^b \int_c^d \int_0^y \int_\Omega k_2(x,y,z,s) u_{nm}(z,s) v_{nm}(x,y) w(x,y) dzdsdxdy$$

Is inner product, the proximate solution is

$$u_{nm}(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} \beta_{ij} \left[\varphi_i(x)\varphi_j(y) + \mu_{ij}\left(\varphi_{i+1}(x)\varphi_{j+1}(y)\right)\right] \quad ; \quad (x,y) \in [0,1] \times [0,1]$$

Where the control point β_{ij} are undermined constants coefficients, μ_{ij} is constant chosen as the initial conditions and Laguerre polynomial $\varphi_i(x), \varphi_j(y)$; $i = 0,1,2,\cdots,n$; $j = 0,1,2,\cdots,m$ are used.

From Eqs. (4.2) we obtain the following result

$$\sum_{i=0}^{n} \sum_{j=0}^{m} \beta_{ij}\, (\varphi_i(x)\varphi_j(y) + \mu_{ij}(\varphi_{i+1}(x)\varphi_{j+1}(y)), \xi_r(x)\xi_\tau(y))_w = (f, \xi_r(x)\xi_e(y))_w$$
$$+ \sum_{i=0}^{n} \sum_{j=0}^{m} \beta_{ij} [K_1\, (\varphi_i(z)\varphi_j(s) + \mu_{ij}(\varphi_{i+1}(z)\varphi_{j+1}(s))), \xi_r(x)\xi_\tau(y)]_w, \quad (4.3a)$$

or

$$\sum_{i=0}^{n} \sum_{j=0}^{m} \beta_{ij}\, (\varphi_i(x)\varphi_j(y) + \mu_{ij}(\varphi_{i+1}(x)\varphi_{j+1}(y)), \xi_r(x)\xi_\tau(y))_w = (f, \xi_r(x)\xi_e(y))_w$$
$$+ \sum_{i=0}^{n} \sum_{j=0}^{m} \beta_{ij} [K_2\, (\varphi_i(z)\varphi_j(s) + \mu_{ij}(\varphi_{i+1}(z)\varphi_{j+1}(s))), \xi_r(x)\xi_\tau(y)]_w \quad (4.3b)$$

Where $\xi_r(x), \xi_\tau(y), r = 0,1,\cdots,n\,, e = 0,1,\cdots,m$ are chepyshev polynomial test function corresponding to the weigh function $w(x,y) = \frac{1}{\sqrt{(1-x^2)(1-y^2)}}$, with $i = 0,1,\cdots,n\,, j = 0,1,\cdots,m$.
From Eqs. (4.3) can be get the following matrix forms

$$(A + B)U_{ij} = F \quad (4.4)$$

where,

$$U_{ij} = [u_0u_0, \cdots, u_ou_m, u_1u_o, \cdots, u_1u_m, \cdots, u_nu_0, \cdots, u_nu_m]^T \quad (4.5)$$

$$A(a_{ijr\tau})_{i,j,r,\tau=0}^{nm} = (\varphi_i(x)\varphi_j(y) + \mu_{ij}(\varphi_{i+1}(x)\varphi_{j+1}(y)), \xi_r(x)\xi_\tau(y))_w \quad (4.6)$$

$$B(b_{ijr\tau})_{i,j,r,\tau=0}^{nm} = (K_\ell(\varphi_i(z)\varphi_j(s) + \mu_{ij}(\varphi_{i+1}(z)\varphi_{j+1}(s))), \xi_r(x)\xi_\tau(y))_w, \quad \ell = 1,2 \quad (4.7)$$

$$F(f_{r\tau})_{r,\tau=0}^{nm} = \left(f, \xi_r(x)\xi_\tau(y)\right)_w \quad (4.8)$$

5. Numerical Examples

An overall analysis of SPGM performance for a 2D IE solution is based on the absolute error values from different methods. In other words, by comparing the results for different numbers of n and m; the outcome will give us a clear idea about how correctly and effectively this method works, explaining its faithful applicability to similar techniques like 2D-triangular functions [2], rationalized Haar functions (RHF) [5], successive approximations method (SAM) [15], 2D-orthonormal Bernstein collocation method (2D-OBPs) [16], Bernoulli collocation method (BCM) [11], and discretization methods [12].

Example 5.1: Consider the following linear VIE in 2D [2]:

$$u(x,y) = f(x,y) - \int_0^x \int_0^y \sin(x - y - z + s)\, u(z,s)dzds,$$

where $f(x,y) = (x - y + 1)cos(x + y) - (x - y)cos(x - y)$. The exact solution $u(x,y) = cos(x + y)$**.**

The numerical results are compared with the numerical results obtained by using 2D- triangular functions [2] in Tables 5.1-5.3. The absolute error, approximate solution and exact solution for $n = m = 4$ are plotted in Figure 5.1.

Example 5.2: Consider the following linear MIE in 2D [13]:

$$u(x,y) = f(x,y) + \int_0^x \int_0^1 (x + s)u(z,s)dzds$$

where $f(x,y) = xy - \frac{5}{12}x^3$. The exact solution $u(x,y) = xy$**.**

The numerical results are compared with the numerical results obtained by using Bernoulli collocation method (BCM) [11], successive approximations method (SAM) [15] and rationalized Haar functions (RHF) [5] in Tables 5.4-5.6. The absolute error, approximate solution and exact solution for $n = m = 4$ is are plotted in Figure 5.2.

Example 5.3: Consider the following linear MIE in 2D [18]:

$$u(x,y) = f(x,y) + \int_0^x \int_0^1 (2s - 1)e^z u(z,s)dzds$$

where $f(x,y) = sin(x) + y - \frac{1}{6}e^x + \frac{1}{6}$. the exact solution $u(x,y) = sin(x) + y$**.** The numerical results are compared with the numerical results obtained by using 2D-orthonormal Bernstein collocation method (2D-OBPs) [16], Bernoulli collocation method (BCM) [11] in Table 5.7. The absolute error, approximate solution and exact solution for $n = m = 8$ is are plotted in Figure 5.3.

Example 5.4: Consider the following linear VIE in 2D [12]:

$$u(x,y) = f(x,y) + \int_0^y \int_0^x (xz + \cos s)u(z,s)dzds$$

where$f(x,y) = x \sin y - \frac{1}{4}x^5 + \frac{1}{4}x^5 \cos y - \frac{1}{4}x^2 \sin^2 y$. The exact solution $u(x,y) = x \sin y$.

The numerical results are compared with the numerical results obtained by using discretization method [12] Table 5.8. The absolute error, approximate solution and exact solution for $n = m = 8$ is are plotted in Figure 5.4.

Table 5.1: Absolute errors at test points (x,y) with $n = m = 4$ for Example 5.1

$(\boldsymbol{x},\boldsymbol{y})$	***SPGM***	**[3]**
(0,0)	8×10^{-5}	0
(0.1,0.1)	2×10^{-5}	1.4575×10^{-2}
(0.2,0.2)	1.4×10^{-4}	9.3562×10^{-3}
(0.3,0.3)	3.1×10^{-4}	1.7768×10^{-3}
(0.4,0.4)	4×10^{-4}	4.2834×10^{-3}
(0.5,0.5)	3.1×10^{-4}	4.8953×10^{-4}
(0.6,0.6)	0	3.6187×10^{-2}
(0.7,0.7)	4.3×10^{-4}	2.8238×10^{-2}
(0.8,0.8)	7.58×10^{-4}	5.0821×10^{-2}
(0.9,0.9)	5.4×10^{-4}	7.8802×10^{-2}

Table 5.2: Absolute errors at test points (x,y) with $n = m = 8$ for Example 5.1

$(\boldsymbol{x},\boldsymbol{y})$	***SPGM***	**[3]**
(0,0)	0	0
(0.1,0.1)	0	2.45×10^{-3}
(0.2,0.2)	0	1.59×10^{-3}
(0.3,0.3)	0	2.39×10^{-3}
(0.4,0.4)	0	5.32×10^{-3}
(0.5,0.5)	0	3.00×10^{-4}
(0.6,0.6)	0	1.03×10^{-2}
(0.7,0.7)	0	2.23×10^{-2}
(0.8,0.8)	0	2.91×10^{-2}
(0.9,0.9)	0	2.41×10^{-2}

Table 5.3: Absolute errors at test points (x, y) with $n = m = 16$ for Example 5.1

(x, y)	***SPGM***	**[3]**
(0,0)	0	0
(0.1,0.1)	0	6.85×10^{-4}
(0.2,0.2)	1×10^{-5}	3.47×10^{-4}
(0.3,0.3)	2×10^{-5}	9.94×10^{-4}
(0.4,0.4)	3×10^{-5}	3.88×10^{-3}
(0.5,0.5)	3×10^{-5}	7.68×10^{-5}
(0.6,0.6)	5×10^{-5}	8.37×10^{-3}
(0.7,0.7)	5×10^{-5}	7.65×10^{-3}
(0.8,0.8)	4.3×10^{-5}	8.70×10^{-3}
(0.9,0.9)	4×10^{-5}	1.5×10^{-2}

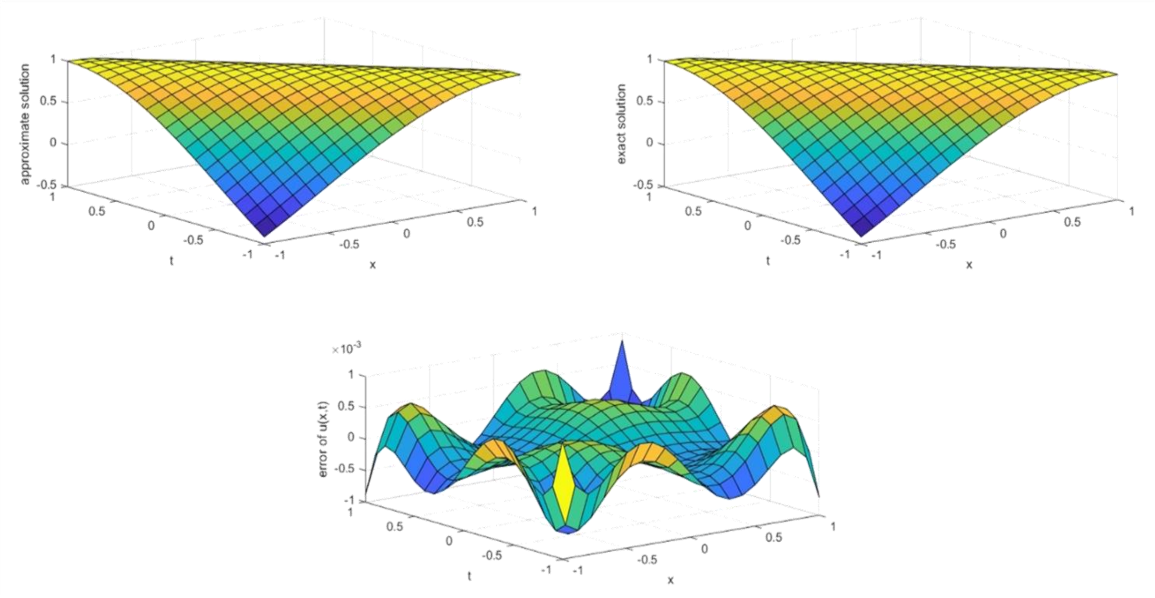


Figure 5.1: Plot of the exact solution, approximate solution and max. error for Example 5.1

Table 5.4: Absolute errors at test points (x, y) with $n = m = 3$ for Example 5.2

(x, y)	**SPGM**	BCM **[11]**
(0,0)	7.1686×10^{-15}	3.2969690×10^{-1}
(0.1,0.1)	0	3.2760781×10^{-1}
(0.2,0.2)	0	3.1576020×10^{-1}
(0.3,0.3)	0	2.9536746×10^{-1}
(0.4,0.4)	0	2.6797218×10^{-1}
(0.5,0.5)	0	2.3534264×10^{-1}
(0.6,0.6)	0	1.9937474×10^{-1}
(0.7,0.7)	0	1.6199934×10^{-1}
(0.8,0.8)	0	1.250947×10^{-1}
(0.9,0.9)	0	9.040599×10^{-2}

Table 5.5: Absolute errors at test points (x, y) with $n = m = 4$ for Example 5.2

$(\boldsymbol{x}, \boldsymbol{y})$	***SPGM***	***SAM*** **[15]**
(0,0)	2.5698×10^{-3}	5.196012×10^{-1}
(0.1,0.1)	3.002×10^{-3}	5.213247×10^{-1}
(0.2,0.2)	3.502×10^{-3}	5.185562×10^{-1}
(0.3,0.3)	4.093×10^{-3}	5.164139×10^{-1}
(0.4,0.4)	4.81×10^{-3}	5.141450×10^{-1}
(0.5,0.5)	5.68×10^{-3}	5.126711×10^{-1}
(0.6,0.6)	6.74×10^{-3}	5.102234×10^{-1}
(0.7,0.7)	8.05×10^{-3}	5.083669×10^{-1}
(0.8,0.8)	9.65×10^{-3}	5.063321×10^{-1}
(0.9,0.9)	1.163×10^{-2}	5.044418×10^{-1}

Table 5.6: Absolute errors at test points (x, y) with $n = m = 18$ for Example 5.2

$\boldsymbol{x}, \boldsymbol{y})$	***SPGM***	***RHF*** **[5]**
(0,0)	8.72×10^{-17}	2.251543×10^{-1}
(0.1,0.1)	0	2.159587×10^{-1}
(0.2,0.2)	0	2.085641×10^{-1}
(0.3,0.3)	0	1.94871×10^{-1}
(0.4,0.4)	0	1.665021×10^{-1}
(0.5,0.5)	0	1.498517×10^{-1}
(0.6,0.6)	0	1.422590×10^{-1}
(0.7,0.7)	0	1.393651×10^{-1}
(0.8,0.8)	0	1.321355×10^{-1}
(0.9,0.9)	0	1.272933×10^{-1}

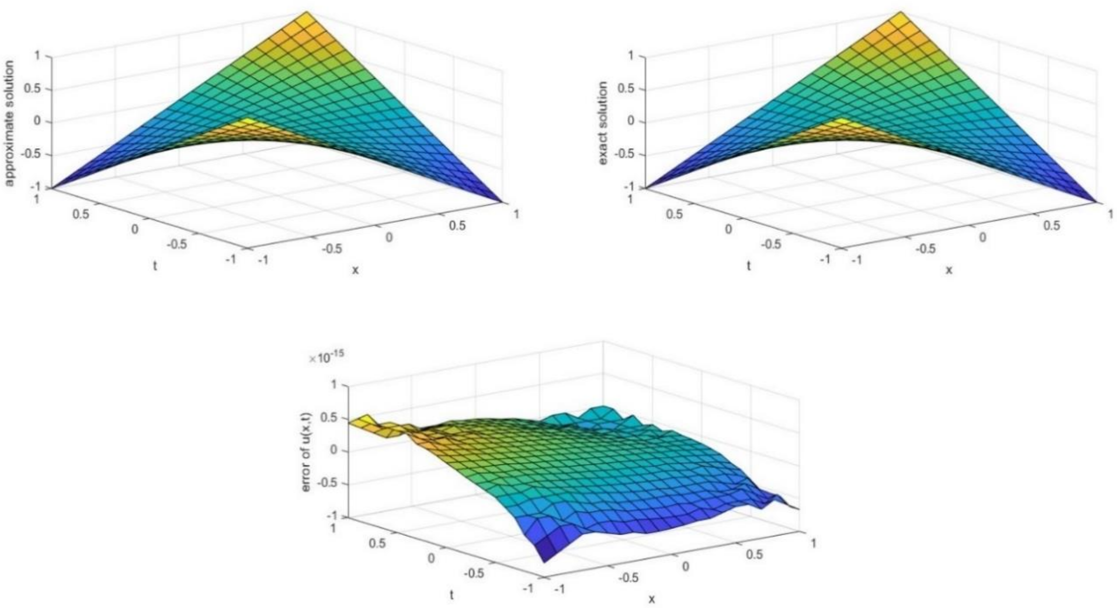
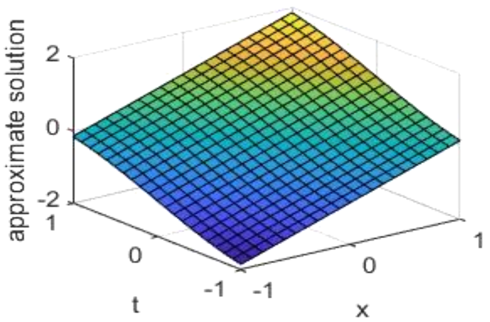
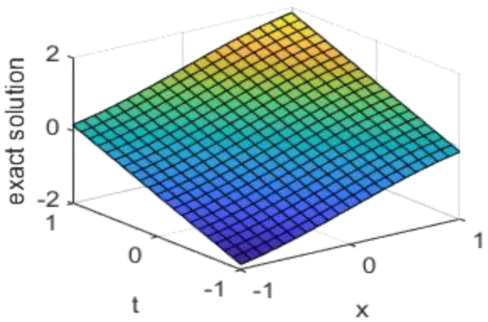


Figure 5.2 : Plot of the exact solution , approximate solution and max. error for Example 5.2

Table 5.7: Absolute errors at test points (x, y) with $n = m = 8$ for Example 5.3

(x, y)	$SPGM$	BCM [11]	$2D - OBPs$ [16]
(0,0)	1.29×10^{-14}	0	1.7590×10^{-10}
(0.1,0.1)	0	6.73360×10^{-8}	6.3487×10^{-12}
(0.2,0.2)	0	1.08210×10^{-7}	3.6477×10^{-11}
(0.3,0.3)	0	1.07410×10^{-7}	2.0662×10^{-10}
(0.4,0.4)	0	6.59240×10^{-8}	8.0123×10^{-10}
(0.5,0.5)	0	1.3766×10^{-14}	2.69990×10^{-9}
(0.6,0.6)	0	16.52040×10^{-8}	7.74040×10^{-9}
(0.7,0.7)	0	1.05080×10^{-7}	1.95510×10^{-8}
(0.8,0.8)	0	1.04700×10^{-7}	4.46400×10^{-8}
(0.9,0.9)	0	6.44420×10^{-8}	9.34810×10^{-8}





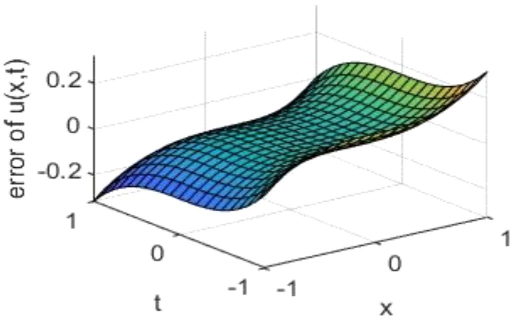


Figure 5.3 : Plot of the exact solution , approximate solution and max. error for Example 5.3

Table 5.8: Absolute errors at test points (x, y) for Example 5.4

(x, y)	***SPGM*** n=m=6	***SPGM*** n=m=8	***[12]*** n=m=6
(0,0)	2.1808×10^{-14}	1.9902×10^{-13}	2.38053×10^{-11}
(0.2,0.2)	1×10^{-6}	0	4.24890×10^{-9}
(0.4,0.4)	0	0	3.25418×10^{-9}
(0.6,0.6)	0	0	4.26163×10^{-9}
(0.8,0.8)	0	0	1.98141×10^{-8}
(1,1)	0	0	2.69535×10^{-8}

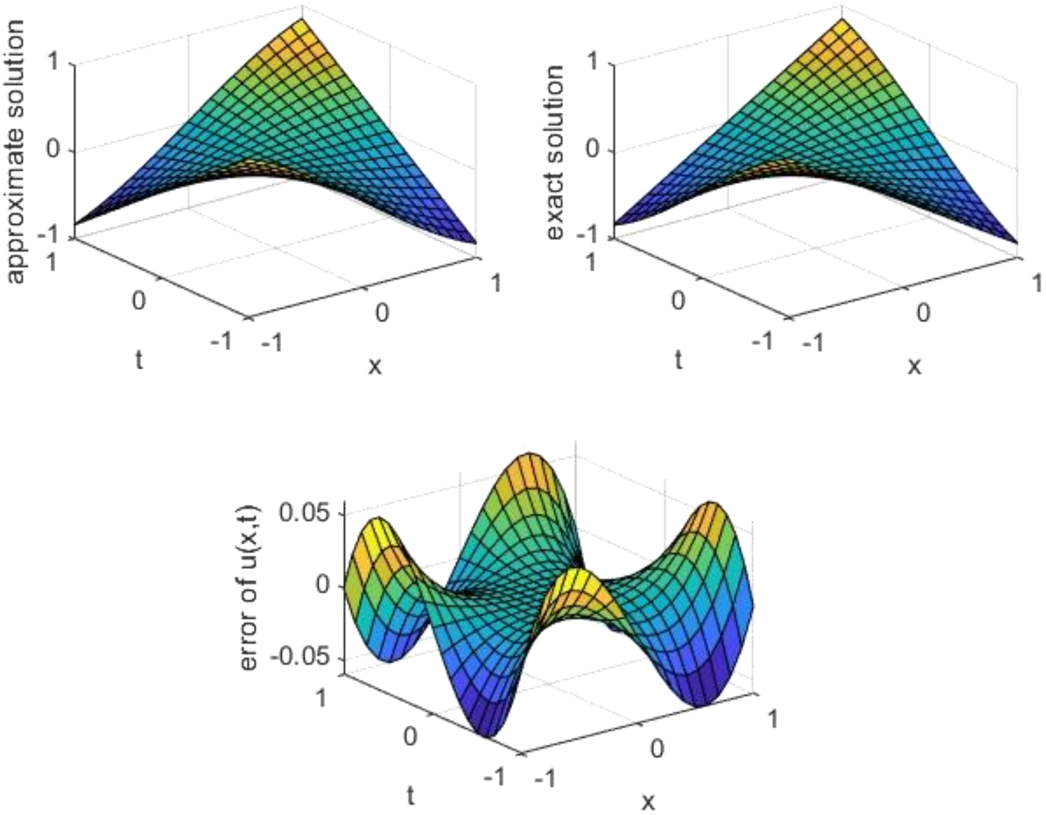


Figure 5.4: Plot of the exact solution, approximate solution and max. error for Example 5.4

6. Conclusion

In this work, a spectral Petrov-Galerkin method for solving two-dimensional linear integral equations (Volterra and mixed Volterra-Fredholm) was introduced. The present method converts the integral equations into matrices, namely blocks. Moreover, it should be noted that in the first example of the Volterra equation, a comparison was made between the outcomes of this method with the two-dimensional method of triangular functions [3]. It was seen that the absolute error is more accurate, as shown in Tables 5.1-5.3 and Figure 5.1. Similarly, comparisons were made with the Bernoulli approximation method [11], the successive approximation method [15], and the rationalized Haar function method [5] for the mixed Volterra-Fredholm equation. The comparison revealed a minimum absolute error using the new method, as shown in Tables 5.4-5.6 and Figure 5.2. The new method was also compared with the two-dimensional orthogonal Bernstein approximation method [16] for another mixed equation. This has shown a higher level of accuracy, as shown in Table 5.7 and Figure 5.3. Moreover, comparisons were made with the discretization method [12] for the fourth example of the Volterra equation, where the new method showed a minimum absolute error, as shown in Table 5.8 and Figure 5.4. The results illustrated in Tables 5.1-5.8 corroborate the efficiency of the method in solving two-dimensional functions of integral equations.

7. References

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طريقة بيتروف-كَالركن الطيفية لحل المعادلات التكاملية ثنائية الأبعاد من النوع الثاني

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الملخص

بأستخدام طريقة بيتروف-جالركن الطيفي (SPGM)، فأن الهدف الرئيسي من هذا البحث هو اقتراح حل عددي لمعادلات فولتيرا الخطية ثنائية الأبعاد ومعادلات فولتيرا-فريدهولم التكاملية المختلطة (VIEs وMIEs، على التوالي). وذلك بحل أربع أمثلة حيث توضح نتائج الطريقة مقارنةً بالطرق الاخرى من ناحية الدقة والخطأ فهي طريقة قوية لحل هذا النوع من المعادلات.

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