



An Improvement of the Order of Approximation by the Sequence of Bernstein-Kantorovich Operators

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ABSTRACT

This paper presented an improvement of the order of approximation by the sequence of Bernstein-Kantorovich type operators in three ways. The approximation order obtained by these sequences is $O(n^{-1})$, $O(n^{-2})$ and $O(n^{-3})$ respectively. Some theoretical results related to the convergence theorem and Voronovskaja asymptotic formula of the improvement sequences are presented. Then, some numerical examples for these sequences are given. The numerical results are supported by the improvement of the order of approximation. These improvements are done based on the research idea [9].

1. Introduction.

The very famous classical Bernstein sequence which was given in (1912) by Bernstein is defined as [2,12,16]

$$B_n(f; x) = \sum_{k=0}^n b_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \dots, n, \quad x \in [0, 1].$$

Many researchers have tried to improve the approximation of the order by a sequence of classical Bernstein $O(n^{-1})$ to a better order by defining some modifications of the classical Bernstein sequence. Unfortunately, many of these modifications were given have the same order as the classical Bernstein sequence. On the other side, some new techniques are introduced for this purpose like Micchelli and the linear combinations which improve the order approximation But, these techniques have no good applications because they have more arithmetic operations and more complicated algorithms in programming difficulties [1,8,14].

Khosravian, Dehghan, and Eslahchi submitted an enhancement of the order approximation with the alteration of a sequence of conventional Bernstein operations [9].

Shehab presented an improved approximation of the order of another modification of a sequence of Bernstein operators as follows:

$$B_{n,r}(f; x) = \sum_{k=0}^n b_{n,k}(x) f\left(x + \left(\frac{k}{n} - x\right)^r\right), \quad x \in [0, 1],$$

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where $r = \{1,3, \dots\}$ they are odd positive integers and prove the Voronovskaja-type an asymptotic formula when ($r = 3$) giving us the order of approximation $O(n^{-2})$ which is better than the order of the classical Bernstein sequences $O(n^{-1})$ [10,11].

Kantorovich (1959) is given a modification to the polynomials $B_n(f; x)$ as [13,15],

$$KB_n(f; x) = (n + 1) \sum_{k=0}^n b_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, x \in [0,1].$$

This paper presented an improvement of the order of approximation of a modification by the classical Bernstein-Kantorovich sequence is defined as the following

$$KB_n(f; x) = (n + 1) \sum_{k=0}^n b_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, x \in [0,1].$$

2. Main Results.

This paper, presented an improvement of the order of approximation of a modification by the classical sequence Bernstein-Kantorovich operators is defined as the following:

$$KB_n^{M,1}(f; x) = (1 + n) \sum_{k=0}^n KB_{n,k}^{M,1}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt$$

$$KB_n^{M,1}(f; x) = (1 + n) \sum_{k=0}^n \left((1 - x)b_{n-1,k}(x) + xb_{n-1,k-1}(x) \right) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \tag{1}$$

$$= (1 - x) \left((1 + n) \sum_{k=0}^n b_{n-1,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \right)$$

$$+ x \left((1 + n) \sum_{k=0}^n b_{n-1,k-1}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \right)$$

$$:= (1 - x)\Psi_1(f; x) + x\Psi_2(f; x),$$

where

$$KB_{n,k}^{M,1}(x) = b_{n-1,k}(x)\dot{a}(x, n) + b_{n-1,k-1}(x)\dot{a}(1 - x, n), \tag{2}$$

and $\dot{a}(x, n) = a_1(n) + a_2(n)x$ where $a_i(n), n = 0,1,2, \dots, i = 1,2$. By finding the conditions of Korovkin's theorem it is possible to know the find form.

Now, to obtain these unknown sequences under the conditions that fit the function f . The Voronovskaja-type asymptotic formula for the sequence $KB_n^{M,1}(f; x)$ is generally has the form,

$$\lim_{n \rightarrow \infty} n (KB_n^{M,1}(f; x) - f(x)) = C_1(x)f(x) + C_2(x)f'(x) + C_3(x)f''(x), \tag{3}$$

where $C_1(x), C_2(x)$ and $C_3(x)$ are independent of n .

Suppose that $e_i(t) = t^i$, where $i = 0, 1, 2$, and from Korovkin theorem one can get that

$$KB_n^{M,1}(e_0; x) = a_2(n) + 2a_1(n). \tag{4}$$

From which we get it,

$$\Psi_1(e_0; x) = \Psi_2(e_0; x) = 1. \tag{5}$$

Well by finding,

$$\Psi_1(e_1; x) = \left[\frac{n - 1}{n} x \right], \quad \Psi_2(e_1; x) = \left[\frac{1}{n} + \frac{n - 1}{n} x \right]. \tag{6}$$

and

$$\begin{aligned} \Psi_1(e_2; x) &= \left[\frac{n-1}{n^2} x + \frac{(n-1)(n-2)}{n^2} x^2 \right], \\ \Psi_2(e_2; x) &= \left[\frac{1}{n^2} + 3 \frac{n-1}{n^2} x + \frac{(n-1)(n-2)}{n^2} x^2 \right]. \end{aligned} \tag{7}$$

Thus, we get the following results,

$$\begin{aligned} K\mathcal{B}_n^{M,1}(e_1; x) &= a(x, n) \left[\frac{n-1}{n} x \right] + a(1-x, n) \left[\frac{1}{n} + \frac{n-1}{n} x \right] \\ &= x(a_2(n) + 2a_1(n)) + \left[\frac{(1-2x)(a_2(n) + a_1(n))}{n} \right]. \end{aligned} \tag{8}$$

Similarly, after doing some math steps, we find that,

$$\begin{aligned} K\mathcal{B}_n^{M,1}(e_2; x) &= x^2(a_2(n) + 2a_1(n)) + \left[\frac{(4x - 6x^2)a_1(n) + (3x - 5x^2)a_2(n)}{n} \right] \\ &\quad + \left[\frac{(1 - 4x + 4x^2)(a_2(n) + a_1(n))}{n^2} \right] \end{aligned} \tag{9}$$

If the relation (4) is verified, one has got the equation,

$$a_2(n) + 2a_1(n) = 1.$$

To evaluate the unknown sequences above, one can follow the following cases:

- Case 1. If $a_2(n) = 0$, then $a_1(n) = \frac{1}{2}$,
- Case 2. If $0 < a_2(n) < 1$, then $a_1(n) > 0$,
- Case 3. If $a_2(n) = 1$, then $a_1(n) = 0$,
- Case 4. If $a_2(n) > 1$, then $a_1(n) < 0$,
- Case 5. If $-1 < a_2(n) < 0$, then $a_1(n) > 0$ and $a_2(n) + a_1(n) > 0$,
- Case 6. If $a_2(n) = -1$, then $a_1(n) = 1$,
- Case 7. If $a_2(n) < -1$, then $a_1(n) > 0$ and $a_2(n) + a_1(n) < 0$.

We can now analyze the unknown sequences $a_2(n)$, $a_1(n)$ easily and it is easy to investigate (1.2) case 6 ($a_2(n) = -1, a_1(n) = 1$) of the sequence (1) is reduced. Like that, we can easily verify operator (2) is verified, except for case 4 and case 7.

Theorem 1.

Suppose that $a_2(n), a_1(n)$ a sequence in the sequence (2) is verified. If $f \in [0,1]$,
 $\Rightarrow \lim_{n \rightarrow \infty} nK\mathcal{B}_n^{M,1}(f; x) = f(x)$ is uniformly on $[0,1]$.

Proof: By using that at $a_i(n)$, $i = 1, 2$ in instances 6,5,3,2,1. Are acceptable, so, $|(a_i(n))| \leq M$, also through interest from (8) and (9) we get,

$$\lim_{n \rightarrow \infty} nK\mathcal{B}_n^{M,1}(e_j; x) = e_j \text{ on } [0,1] \text{ is uniformly.}$$

where $j = 0, 1, 2$.

Hence, the conditions of Korovkin's theorem are held, and the required subsequence is held.

■

Theorem 2.

For $f \in C[0,1]$ and for every $a_2(n), a_1(n)$ are convergence sequences hold of case 4 and case 7, we get,

$$\lim_{n \rightarrow \infty} K\mathcal{B}_n^{M,1}(f; x) = f(x) \text{ on } [0,1] \text{ is uniformly.}$$

Proof. Let $a_2(n) > 1$ and $a_1(n) < 0$. we will suppose that,

$$T_{n,1}(f; x) = (1+n) \sum_{k=0}^n \left(x b_{n-1,k}(x)(a_2(n)) + b_{n-1,k-1}(x)(a_2(n)) \right) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt,$$

and

$$T_{n,2}(f; x) = (n+1) \sum_{k=0}^n \left(-(a_1(n)x)b_{n-1,k}(x) + (a_2(n)x - a_1(n))b_{n-1,k-1}(x) \right)$$

$$\times \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f\left(\frac{k}{n}\right) dt$$

These two sequences can be easily proven $T_{n,2}(f; x)$, $T_{n,1}(f; x)$, that they are correct Also in addition to that we can check that,

$$KB_n^{M,1}(f; x) = T_{n,1}(f; x) - T_{n,2}(f; x).$$

Well, now we will get the following facts,

$$\begin{aligned} T_{n,1}(e_0; x) &= a_2(n)(1 + x), \\ T_{n,1}(e_1; x) &= a_2(n)(1 + x) \left[\frac{(n-1)}{n} x \right] + \left[\frac{1}{n} \right] a_2(n), \end{aligned} \tag{10}$$

$$\begin{aligned} &T_{n,1}(e_2; x) \\ = (x + 1)(a_2(n)) &\left[\frac{(n-2)(n-1)}{n^2} x^2 \right] + \left[\frac{1 + (x^2 + 3x)(n-1)}{n^2} \right] (a_2(n)), \end{aligned} \tag{11}$$

and

$$\begin{aligned} T_{n,2}(e_0; x) &= a_2(n)x - 2a_1(n), \\ T_{n,2}(e_1; x) &= (a_2(n)x - 2a_1(n)) \left[\frac{(n-1)}{n} x \right] + \left[\frac{a_2(n)x - a_1(n)}{n} \right], \end{aligned} \tag{12}$$

$$\begin{aligned} T_{n,2}(e_2; x) &= (a_2(n)x - 2a_1(n)) \left[\frac{(n-1)(n-2)}{n^2} x^2 \right] \\ &+ \left[\frac{(3(n-1)x + 1)a_2(n)x - (4(n-1)x + 1)a_1(n)}{n^2} \right]. \end{aligned} \tag{13}$$

Through interest from extended Korovkin's Theorem and (10)- (13), we get,

$$\lim_{n \rightarrow \infty} T_{n,1}(f; x) = (f(x)H_1(x + 1)) \text{ on } [0,1] \text{ is uniformly.}$$

$$\lim_{n \rightarrow \infty} [T_{n,2}(f; x)] = (f(x)(H_1(x + 1) - 1)) \text{ on } [0,1] \text{ is uniformly.}$$

And that $\lim_{n \rightarrow \infty} a_2(n) = H_1$, and so, we obtained the required results, through case 7, and in the same way ends immediately. ■

Now, we can develop the results we will have a formula for a second-degree approximation. Because of that, we will modify the sequence of Bernstein-Kantorovich operators in (1)-(2), as,

$$KB_n^{M,2}(f; x) = (1 + n) \sum_{k=0}^n KB_{n,k}^{M,2}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt \tag{14}$$

where

$$KB_{n,k}^{M,2}(x) = b_{n-2,k}(x)\dot{b}(x, n) + b_{n-2,k-2}(x)\dot{b}(1 - x, n) + b_{n-2,k-1}(x)\dot{d}(x, n), \tag{15}$$

and

$$\dot{b}(x, n) = x^2 b_3(n) + x b_2(n) + b_1(n), \quad \dot{d}(x, n) = x(1 - x) b_4(n), \tag{16}$$

and $b_i(n)$, $i = 1, 2, 3, 4$, they will be challenged later because they are unknown sequences. Point out that of $b_1(n) = b_3(n) = 1$, $b_2(n) = -2$, $b_4(n) = 2$, operator (14), (15), and (16) reduces to the sequences of Bernstein- Kantorovich operators (1) - (2).

We will discuss the first three special cases of the function f to explain the following results, through which is following theorems are proved. Suppose that $e_i(t) = t^i$, where $i = 0, 1, 2$, and from Korovkin's theorem one can get that,

$$KB_n^{M,2}(e_0; x) = \dot{b}(x, n) + \dot{b}(1 - x, n) + \dot{d}(x, n), \tag{17}$$

which can be obtained from the fact,

$$\Psi_1(e_0; x) = 1, \Psi_2(e_0; x) = 1 = \Psi_3(e_0; x).$$

Now, we find at $f(t) = t^1$, we get,

where

$$\sum_{k=0}^n b_{n-2,k}(x) = \sum_{k=0}^n b_{n-2,k-1}(x) = \sum_{k=0}^n b_{n-2,k-2}(x) = 1.$$

So, we find at $f(t) = t^2$, we get,

$$KB_n^{M,2}(e_1; x) = b(x, n) \left[\frac{(n-2)}{n} x \right] + d(x, n) \left[\frac{1}{n} + \frac{(n-2)}{n} x \right] + b(1-x, n) \left[\frac{2}{n} + \frac{(n-2)}{n} x \right] \tag{18}$$

So, by the same style, we get,

$$\begin{aligned} KB_n^{M,2}(e_2; x) &= \frac{1}{n} KB_n^{M,2}(t; x) + KB_n^{M,2}(t; x) \left(t \left(t - \frac{1}{n} \right); x \right) = \\ &= \frac{1}{n} \left(b(x, n) \left[\frac{(n-2)}{n} x \right] + d(x, n) \left[\frac{1}{n} + \frac{(n-2)}{n} x \right] + b(1-x, n) \left[\frac{2}{n} + \frac{(n-2)}{n} x \right] \right) \\ &+ b(x, n) \left[\frac{(n-3)(n-2)}{n^2} x^2 \right] + d(x, n) \left[\frac{2(n-2)}{n^2} x + \frac{(n-3)(n-2)}{n^2} x^2 \right] \\ &+ b(1-x, n) \left[\frac{2}{n^2} + \frac{4(n-2)}{n^2} x + \frac{(n-3)(n-2)}{n^2} x^2 \right] \\ &= b(x, n) \left[\frac{(n-2)}{n^2} x + \frac{(n-3)(n-2)}{n^2} x^2 \right] \\ &+ d(x, n) \left[\frac{1}{n^2} + \frac{3(n-2)}{n^2} x + \frac{(n-3)(n-2)}{(n+1)^2} x^2 \right] \\ &+ b(1-x, n) \left[\frac{4}{n^2} + \frac{5(n-2)}{n^2} x + \frac{(n-3)(n-2)}{(n+1)^2} x^2 \right], \end{aligned} \tag{19}$$

Next, will define sequences and related results $b_i(n), i = 1, 2, 3, 4$, to get the operator of the new quadratic approximation operator, it will use the values obtained for $i = 0, 1, 2$. $KB_n^{M,2}(e_i; x)$ (from (17), (18) with (19)), then, $KB_n^{M,2}(e_0; x) = 1$, we show,

$$b_4(n) = 2b_3(n), \quad b_3(n) + b_2(n) + 2b_1(n) = 1, \text{ as } n \rightarrow \infty. \tag{20}$$

By (20) and the use of (18), we get,

$$KB_n^{M,2}(e_1; x) = \left[\frac{2(1-b_1(n))}{n} \right] + \left[\frac{(-4(1-b_1(n)+n)x}{n} \right]. \tag{21}$$

Now, if we set $b_1(n) = 1$, then from (20) and (21) we obtain,

$$\begin{aligned} KB_n^{M,2}(e_2; x) &= x, \\ b_2(n) &= -1 - b_3(n), \quad d_1(n) = -2(1 + b_2(n)). \end{aligned} \tag{22}$$

Substituting (20) and (22) into (19) will result,

$$KB_n^{M,2}(e_2; x) = x^2 - \left[\frac{x(2b_3(n) - 2b_3(n)x - 2 + xn - n + 2x)}{n^2} \right]. \tag{23}$$

To get a second-degree approximation, we make $b_3(n) = \frac{n}{2}$, then (23) reduces to,

$$KB_n^{M,2}(e_2; x) = x^2 + \left[\frac{2x(1-x)}{n^2} \right], \tag{24}$$

and therefore, we get from (22) that,

$$b_2(n) = -1 - \frac{n}{2}, \quad b_4(n) = n.$$

And use the same method. Finally, we got to,

$$KB_n^{M,2}(e_3; x) = x^3 + \left[\frac{2x(5x-1)(1-x)}{n^2} \right] - \left[\frac{6x(2x-1)(1-x)}{n^3} \right], \tag{25}$$

We continue our study for the second scenario, which is the one that follows:

$$b_3(n) = \left(\frac{1}{2} n \right), \quad b_2(n) = -1 - \frac{n}{2}, \quad b_1(n) = 1, \quad b_4(n) = n, \tag{26}$$

with these choices, sequences (14) and (16) we get,

$$K\mathcal{B}_n^{M,2}(f; x) = \sum_{k=0}^n K\mathcal{B}_{n,k}^{M,2}(x) f\left(\frac{k}{n}\right), \tag{27}$$

where

$$K\mathcal{B}_{n,k}^{M,2}(x) = b_{n-2,k}(x) \left[\frac{n}{2}x^2 - \left(1 + \frac{n}{2}\right)x + 1 \right] + b_{n-2,k-1}(x)[nx(1-x)] + b_{n-2,k-2}(x) \left[\frac{n}{2}x^2 - \frac{n}{2}x + x \right]. \tag{28}$$

The following Theorem, through which to impose Voronovskaja formula is close to the operator (27) and (28) that matches the function $f(x)$. Also, in the same theory, we see in operators (27) and (28) that the convergence rate of f is matched with $\mathcal{O}\left(\frac{1}{n^2}\right)$.

Theorem 3.

For $f \in C[0,1]$ and we suppose $x \in [0, 1]$ and $f'''(x)$ exist, so, we get,

$$\lim_{n \rightarrow \infty} n^2 \left(K\mathcal{B}_n^{M,2}(f; x) - f(x) \right) = (x(1-x)) \left(f''(x) + \frac{2x-1}{3} f'''(x) \right).$$

In addition, be correct on $[0,1]$ if $f \in C[0, 1]$. i.e., if $f \in C[0,1]$, also for sufficiently large n , then, we get,

$$K\mathcal{B}_n^{M,2}(f; x) - f(x) = \mathcal{O}\left(\frac{1}{n^2}\right).$$

Where this theory is called Voronovskaja theorem [2,5,7,15].

Now, also for the sequence of Bernstein-Kantorovich operators (1) and (2), will think of another modification, as,

$$K\mathcal{B}_n^{M,3}(f; x) = (1+n) \sum_{k=0}^n K\mathcal{B}_{n,k}^{M,3}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt \tag{29}$$

where

$$K\mathcal{B}_{n,k}^{M,3}(x) = b_{n-4,k}(x)\dot{a}(x,n) + b_{n-4,k-1}(x)\dot{b}(x,n) + b_{n-4,k-2}(x)\dot{c}(x,n) + b_{n-4,k-3}(x)\dot{b}(1-x,n) + b_{n-4,k-4}(x)\dot{a}(x,n), \tag{30}$$

and moreover,

$$\begin{aligned} \dot{a}(x,n) &= x^4c_5(n) + x^3c_4(n) + x^2c_3(n) + xc_2(n) + c_1(n), \\ \dot{b}(x,n) &= x^4c_{10}(n) + x^3c_9(n) + x^2c_8(n) + xc_7(n) + c_6(n), \\ \dot{c}(x,n) &= (x(1-x))^2c_{11}(n). \end{aligned} \tag{31}$$

We see that $c_i(n) = 1, 2, \dots, 11$. Additionally, some sequences that are currently unknown are placed on top of it, and they are chosen so that each operator (29), (30) with (31). Here, too, a new operator will be produced for us, and by taking a third-order approximation, we will notify the operator (29), and

$$\begin{aligned} c_5(n) &= c_1(n) = 1 & c_4(n) &= c_2(n) = -4 \\ c_3(n) &= 6 & c_9(n) &= -c_8(n) = 12 \\ c_6(n) &= 0 & c_{11}(n) &= 6 \end{aligned}$$

Now, we move on to the next theorem using the same technique used to find a second-order operator.

Theorem 4.

Let $f \in C^3[0,1]$. From the operators (29), (30) with (31) and the values for the sequences,

$$\begin{aligned} c_1(n) &= 1 & c_6(n) &= 0 \\ c_3(n) &= 6 + \frac{10}{3}n + \frac{1}{8}n^2 & c_4(n) &= -4 - \frac{14}{3}n - \frac{1}{4}n^2 \\ c_5(n) &= 1 + \frac{23}{12}n + \frac{1}{8}n^2 & c_2(n) &= -4 - \frac{1}{12}n \end{aligned}$$

$$c_{10}(n) = -4 - \frac{23}{3}n - \frac{1}{2}n^2$$

$$c_8(n) = -12 - \frac{31}{3}n - \frac{1}{2}n^2$$

$$c_9(n) = 12 + 17n + n^2$$

$$c_7(n) = 4 + n$$

$$c_{11}(n) = 6 + \frac{23}{2}n + \frac{3}{4}n^2.$$

as $n \rightarrow \infty$. The function f converges uniformly on $[0,1]$. i.e.

$$\lim_{n \rightarrow \infty} KB_n^{M,3}(f; x) = f(x) \text{ on } [0,1] \text{ is uniformly.}$$

Theorem 5.

For $f \in C^3[0,1]$ and we suppose $x \in [0, 1]$ and $f^{(4)}(x)$ exists, so, we get,

$$\lim_{n \rightarrow \infty} [n^3(KB_n^{M,3}(f; x) - f(x))] = f^{(4)}(x) \frac{((1-x)x)^2}{2}.$$

In addition, be correct on $[0,1]$ if $f \in C[0, 1]$. i.e., if $f^{(4)} \in C[0,1]$, also for sufficiently large n , then, we get,

$$KB_n^{M,3}(f; x) - f(x) = o\left(\frac{1}{n^3}\right).$$

by Voronovskaja theorem.

3. Numerical verifications.

Since we are able now to apply the theoretical results that we got from our new operator with some examples numerically. And explain the approximation of the sequence Bernstein-Kantorovich operators, we have chosen the following function, as,

$$f(x) = \begin{cases} 5 - 25x, & 0 \leq x \leq 0.2, \\ 40 \times (0.2 - x)(x - 0.8), & 0.2 \leq x \leq 0.8, \\ 5 - 25 \times (1 - x), & 0.8 \leq x \leq 1. \end{cases}$$

We will symbolize this function in black and the original Bernstein sequence in red $KB_n^{M,0}(f; x)$, the color blue of the sequence of Bernstein-Kantorovich operators of order one $KB_n^{M,1}(f; x)$, the green color of Bernstein-Kantorovich sequences of order two $KB_n^{M,2}(f; x)$ and the color purple of the sequence of Bernstein-Kantorovich operators of order three $KB_n^{M,3}(f; x)$ as in the following:

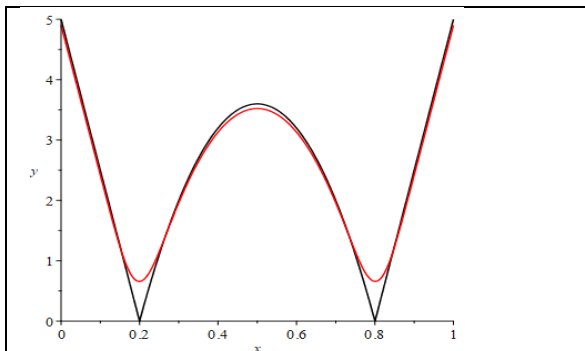


Fig 1. Approximation of $f(x)$ by $KB_n^{M,0}(f;x), n = 128$

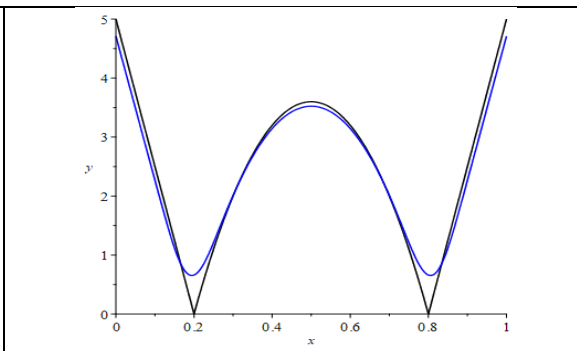


Fig 2. Approximation of $f(x)$ by $KB_n^{M,1}(f;x), n = 128$

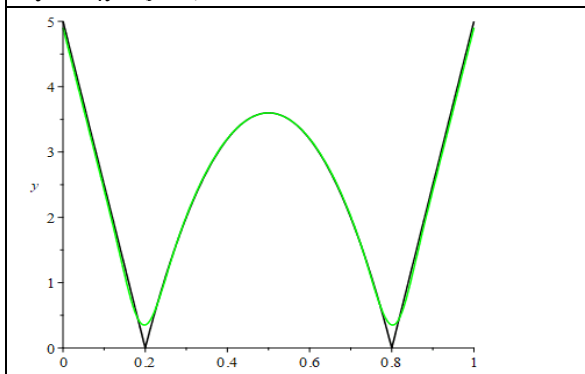


Fig 3. Approximation of $f(x)$ by $KB_n^{M,2}(f;x), n = 128$

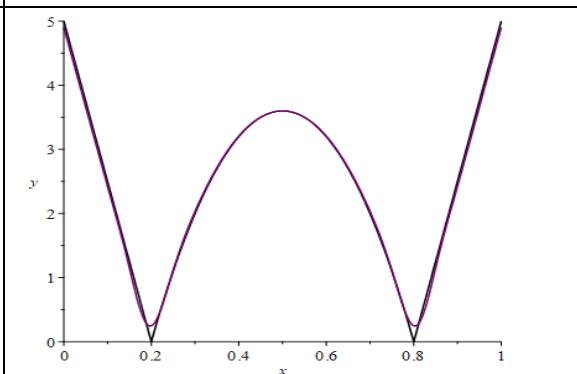


Fig 4. Approximation of $f(x)$ by $KB_n^{M,3}(f;x), n = 128$

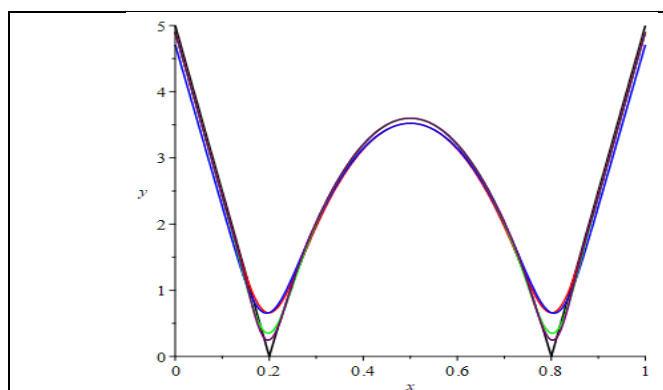


Fig 5. Approximation of $f(x)$ by $KB_n^{M,m}(f;x), m = 0,1,2,3, n = 128$

$$f(x) = \begin{cases} 5 - 25x, & 0 \leq x \leq 0.2, \\ 40(0.2 - x)(x - 0.8), & 0.2 \leq x \leq 0.8, \\ 5 - 25(1 - x), & 0.8 \leq x \leq 1. \end{cases}$$

and the Sequences $KB_n^{M,0}(f;x), KB_n^{M,1}(f;x), KB_n^{M,2}(f;x), KB_n^{M,3}(f;x)$, where $x = [0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1]$. Then take the arithmetic mean of these values for each the sequences.

Table : The Difference Between the Test Function:

n	$K\mathcal{B}_n^{M,0}(f; x)$	$K\mathcal{B}_n^{M,1}(f; x)$	$K\mathcal{B}_n^{M,2}(f; x)$	$K\mathcal{B}_n^{M,3}(f; x)$
5.00	2.122227267	1.923100360	1.909699856	3.908446967
10.00	1.957633071	1.823152692	1.795543817	1.936473788
15.00	1.868330279	1.792123916	1.771250834	1.753244293
20.00	1.827730810	1.781009964	1.767810631	1.747510114
25.00	1.808044757	1.777377205	1.769989305	1.759725698

Conclusion.

This paper theoretically and numerically through the example above has improved the approximation of the order by sequence by the sequence of Bernstein-Kantorovich operators in three ways. The approximation order obtained of orders $O(n^{-1})$, $O(n^{-2})$ and $O(n^{-3})$. We saw how to improve and obtain a better convergence than the original Bernstein sequence, as shown in Fig2, Fig3, and Fig 4. Some theoretical results related to the convergence and the Voronovskaja asymptotic formula of the improvement sequence are presented. Then, some numerical applications of the sequence under study are given. The accuracy of numerical results is proportion with the improvement order, we urge those interested and researchers to consider the new results above because they are much better than the old operators in terms of convergence, accuracy, and speed of implementation.

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نموذج الملخص باللغة العربية لمجلة ابحاث البصرة (العلميات)

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المعلومات البحث	الملخص
الاستلام	12 نيسان 2024
القبول	25 حزيران 2024
النشر	30 حزيران 2024
الكلمات المفتاحية	

قدم هذا البحث تحسين رتبة التقريب من خلال متابعة مؤثر نوع برنستين-كونتروفش بثلاث طرق. الرتب التقريبية الذي تم الحصول عليه بواسطة هذه المتتابعات هو $O(n^{-3}), O(n^{-2}), O(n^{-1})$ على التوالي. تم عرض بعض النتائج النظرية المتعلقة بنظرية التقارب وصيغ فرونسكيا المقاربة لمتتابعات التحسين. ثم تم إعطاء بعض الأمثلة العددية لهذه المتتابعات. يتم دعم النتائج العددية من خلال تحسين رتب التقريب. تتم هذه التحسينات بناءً على فكرة البحث [9].

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