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The Study Focuses on Lacunary Interpolation and Error Estimations by Spline Function

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ABSTRACT

To solve a lacunary interpolation problem with a range of (0, 2, 4), we first look at an appropriate class of insufficient splines. Under suitable assumptions, we prove the existence and uniqueness of the solution. We describe the convergence analysis and show that the approach is of order six and that it can be constructed of any order, with any intermediate lacunary requirements.

1. Introduction

Lacunary interpolation using spline functions is a significant area of study within numerical analysis, particularly for approximating functions and solving differential equations. This method is particularly useful when dealing with incomplete data sets, where certain values are missing (lacunary data).

Lacunary interpolation aims to construct a spline function that approximates a given set of points, even when some of these points are absent. Spline functions, which are piecewise polynomial functions, provide a flexible and smooth approximation that can be tailored to fit the available data points. Recent studies have extended existing methods for lacunary interpolation. For instance, a paper published in 2023 discusses the extension of five-degree spline (0,4) lacunary interpolation on uniform meshes. This research highlights the outcomes, uniqueness, and error boundaries associated

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with this generalized approach, demonstrating its superiority over previous methods using lower-degree splines [1].

Additionally, historical research has laid the groundwork for understanding lacunary interpolation, with early studies dating back to the mid-20th century, focusing on various cases of interpolation and the mathematical frameworks that support these methods. [2-7]

Saeed. R. K. and Karwan H.F. Jwamer [8-10], Phillips [9], and Saeed, R. K [11], examined lacunary interpolation using spline functions of class $C^3[0,1]$. This paper examines a comparable issue. However, the spline function's construction is where the key distinction lies in this instance. To describe it, let's $\nabla: 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a consistent division of the interval $[0,1]$, when $x_j = jh\,, j = 0,1,\ldots,n\,; nh = 1$. The class of spline functions $S_p(6,3,n)$ is defined as follows: { $S_p(6,3,n)$ is the class of spline functions $\mathrm{S}_\nabla(x)$ such that $\mathrm{S}_\nabla(x)$ is a polynomial of degree less than or equal to 6 on each sub-interval $[x_i, x_{i+1}]\,, i = 0,1,\ldots,n-1$ and $\mathrm{S}_\nabla(x) \in C^3[0,1]$ with n knots}. The class of spline functions $\mathrm{Sp}(6,3,\mathrm{n})$ any element $\mathrm{S}_\nabla(x) \in S_p(6,3,n)$ in this work satisfies the following requirements:

$$s_k^{(r)}(x_k) = f^{(r)}(x_k) = y_k^{(r)}.r = 0,2,4\,; k = 0,1,\ldots,n-1, \tag{1}$$

$$s_{n-1}^{(r)}(x_n) = f^{(r)}(x_n) = y_n^{(r)}.r = 0,2,4, \tag{2}$$

$$s_k(x_{k+1}) = s_{k+1}(x_{k+1}) = f(x_{k+1}) = y_{k+1}; k = 0,1,\ldots,n-2, \tag{3}$$

$$s_k^{(r)}(x_{k+1}) = s_{k+1}^{(r)}(x_{k+1}) = f^{(r)}(x_{k+1}) = y_{k+1}^{(r)}\,.r = 0,2,4\,; k = 0,1,\ldots,n-2, \tag{4}$$

$$s_0'(x_0) = f'(x_0) = y_0'\,, s_{n-1}'(x_n) = f'(x_n) = y_n'\,. \tag{5}$$

UNIQUENESS AND EXISTENCE

We now talk about the existence and originality of our approach, which is covered in this section. In the situation of uniform partition and a simplified bounded condition, the analogous error estimate for the much less smooth class of functions $f \in C^6[a,b]$ can be calculated as follows in terms of the modulus of continuity:

Theorem 1: A unique spline function $\mathrm{S}_\nabla(x) \in S_p(6,3,n)$ that meets the requirements 1–5 exists.

Proof: if possible, suppose that

$$\mathrm{S}_\nabla(x) = \begin{cases} \mathrm{S}_0(x) \text{ when } x_0 \le x \le x_1\,, \\ \mathrm{S}_\mathrm{k}(x) \text{ when } x_k \le x \le x_{k+1}, \quad k = 0,1,\ldots,n-2, \\ \mathrm{S}_{\mathrm{n}-1}(x) \text{ when } x_{n-1} \le x \le x_n\,. \end{cases}$$

Then because of the circumstances (1–5). We can write

$$s_0(x) = y_0 + (x-x_0)y_0' + \tfrac{(x-x_0)^2}{2!}y_0'' + (x-x_0)^3\,a_{0,3} + \tfrac{(x-x_0)^4}{4!}y_0^{(4)} + (x-x_0)^5 a_{0,5} + (x-x_0)^6 a_{0,6}\,, \tag{6}$$

$$s_k(x) = y_k + (x-x_k)a_{k,1} + \tfrac{(x-x_k)^2}{2}y_k'' + (x-x_k)^3\,a_{k,3} + \tfrac{(x-x_k)^4}{4!}y_k^{(4)} + (x-x_k)^5 a_{k,5} + (x-x_k)^6 a_{k,6}\,, \tag{7}$$

$$s_{n-1}(x) = y_{n-1} + (x - x_{n-1})a_{n-1,1} + \frac{(x-x_{n-1})^2}{2} y''_{n-1} + (x - x_{n-1})^3\, a_{n-1,3} + \frac{(x-x_{n-1})^4}{4!} y^{(4)}_{n-1}$$
$$+(x - x_{n-1})^5 a_{n-1,5} + (x - x_{n-1})^6 a_{n-1,6}\ , \qquad (8)$$

To get the coefficients in $s_0(x)$, using (3) and (4) with $r = k = 0$. We arrive at the linear system of equations that follows:

$$a_{0,3} + h^2\, a_{0,5} + h^3 a_{0,6} = h^{-3}\left[y_1 - y_0 - hy'_0 - \frac{h^2}{2} y''_0 - \frac{h^4}{24} y^{(4)}_0\right],$$

$$6\, a_{0,3} + 20h^2 a_{0,5} + 30h^3 a_{0,6} = h^{-1}\left[y''_1 - y''_0 - \frac{h^2}{2} y^{(4)}_0\right],$$

$$a_{0,5} + 3h a_{0,6} = \frac{h^{-1}}{120}[y^{(4)}_1 - y^{(4)}_0].$$

After resolving these equations, we obtain:

$$a_{0,3} = h^{-3}\left[\frac{5}{3}(y_1 - y_0) - \frac{5}{3} hy'_0 - h^2\left(\frac{1}{9} y''_1 + \frac{13}{18} y''_0\right) + \frac{h^4}{120}\left(\frac{5}{9} y^{(4)}_1 - \frac{20}{9} y^{(4)}_0\right)\right], \qquad (9)$$

$$a_{0,5} = h^{-5}\left[(y_0 - y_1) + hy'_0 + h^2\left(\frac{1}{6} y''_1 + \frac{1}{3} y''_0\right) - \frac{h^4}{120}\left(\frac{4}{3} y^{(4)}_1 + \frac{11}{3} y^{(4)}_0\right)\right], \qquad (10)$$

$$a_{0,6} = h^{-6}\left[\frac{1}{3}(y_1 - y_0) - \frac{1}{3} hy'_0 - h^2\left(\frac{1}{18} y''_1 + \frac{1}{9} y''_0\right) + \frac{h^4}{120}\left(\frac{7}{9} y^{(4)}_1 + \frac{8}{9} y^{(4)}_0\right)\right]. \qquad (11)$$

To get the coefficients in $s_k(x)$, $k = 0,1,\dots,n-2$, using (1). we arrive at the linear system of equations that follows:

$$a_{k,1} + h^2\, a_{k,3} + h^4 a_{k,5} + h^5 a_{k,6} = h^{-1}\left[y_{k+1} - y_k - \frac{h^2}{2} y''_k - \frac{h^4}{4!} y^{(4)}_k\right],$$

$$6a_{k,3} + 20h^2 a_{k,5} + 30h^3 a_{k,6} = h^{-1}\left[y''_{k+1} - y''_k - \frac{h^2}{2!} y^{(4)}_k\right],$$

$$a_{k,5} + 3h a_{k,6} = \frac{h^{-1}}{120}\left[y^{(4)}_{k+1} - y^{(4)}_k\right].$$

After resolving these equations, we obtain:

$$a_{k,1} + 2a_{k+1,1} = h^{-1}\left[2(y_{k+1} - y_k) + \frac{h^2}{6}(y''_{k+1} - y''_k) - \frac{h^4}{360}\left(25\, y^{(4)}_{k+1} + 65 y^{(4)}_k\right)\right], \qquad (12)$$

$$a_{k,3} = \frac{5}{3} h^{-2} a_{k+1,1} - h^{-3}\left[\frac{5}{3}(y_{k+1} - y_k) + h^2\left(\frac{7}{18} y''_{k+1} + \frac{8}{18} y''_k\right) - \frac{h^4}{120}\left(\frac{130}{9} y^{(4)}_{k+1} + \frac{305}{9} y^{(4)}_k\right)\right] \qquad (13)$$

$$a_{k,5} = h^{-5}\left[(y_{k+1} - y_k) + \frac{h^2}{3}\left(y''_{k+1} + \frac{1}{2} y''_k\right) - \frac{h^4}{360}\left(29 y^{(4)}_{k+1} + 76 y^{(4)}_k\right)\right] - h^{-4} a_{k+1,1}, \qquad (14)$$

$$a_{k,6} = \frac{h^{-5}}{3} a_{k+1,1} - \frac{h^{-6}}{9}\left[3(y_{k+1} - y_k) + h^2\left(y''_{k+1} + \frac{1}{2} y''_k\right) - \frac{h^4}{120}\left(32 y^{(4)}_{k+1} + 73 y^{(4)}_k\right)\right]. \qquad (15)$$

Lastly, applying (2) and (5) to the coefficients of $s_{n-1}(x)$ yields the following:

$$a_{n-1,1} + h^2\, a_{n-1,3} + h^4 a_{n-1,5} + h^5 a_{n-1,6} = h^{-1}[y_n - y_{n-1} - \frac{h^2}{2} y''_{n-1} - \frac{h^4}{4!} y^{(4)}_{n-1}],$$

$$6\, a_{n-1,3} + 20h^2 a_{n-1,5} + 30h^3 a_{n-1,6} = h^{-1}[y''_n - y''_{n-1} - \frac{h^2}{2!} y^{(4)}_{n-1}],$$

$$a_{n-1,5} = \frac{h^{-1}}{120}[y^{(4)}_n - y^{(4)}_{n-1}] - 3h a_{n-1,6}\ .$$

Upon solving these equations, we arrive at:

$$a_{n-1,1} = \frac{h^{-1}}{3}[6(y_n - y_{n-1}) - 3hy'_n + \frac{h^2}{6}(3y''_n - 3y''_{n-1}) - \frac{h^4}{360}\left(3y_n^{(4)} - 3y_{n-1}^{(4)}\right)],$$

$$a_{n-1,3} = \frac{h^{-3}}{6}[10(y_{n-1} - y_n) + 10hy'_n - \frac{h^2}{6}(14y''_n + 16y''_{n-1}) + \frac{h^4}{360}\left(20\, y_n^{(4)} - 50y_{n-1}^{(4)}\right)],$$

$$a_{n-1,5} = h^{-5}\left[(y_n - y_{n-1}) - hy'_n + \frac{h^2}{6}(2y''_n + y''_{n-1}) - \frac{h^4}{360}\left(5\, y_n^{(4)} + 10y_{n-1}^{(4)}\right)\right],$$

$$a_{n-1,6} = \frac{h^{-6}}{3}\left[(y_{n-1} - y_n) + hy'_n - \frac{h^2}{6}(2y''_n + y''_{n-1}) + \frac{h^4}{360}\left(8\, y_n^{(4)} + 7y_{n-1}^{(4)}\right)\right].$$

Because the system of equations' coefficient matrix is a non-singular matrix, we can observe that each of the coefficients as mentioned above has a unique determination. Thus, Theorem 1 has a complete proof.

Error bounds:

An approximation derived from the previous theorem is shown here. The latter provides the method's error-bound result, which is required for proving convergence in the following section. We start with an initial result that can be easily verified through direct computation:

Lemma 1: let $f \in C^6[0,1]$, $e_{k,1} = 2(a_{k,1} - y'_k)$ and $e_{k+1,1} = 2(a_{k+1,1} - y'_{k+1})$

Then: $\left|e_{k+1,1}\right| \le \frac{7(k+1)}{360} h^5 w_6(f;h) \to \left|e_{k,1}\right| \le \frac{7k}{360} h^5 w_6(f;h)$, $k = 0,1,\dots,n-1$.

Proof:

Since $e_{k,1} = 2a_{k,1} - 2y'_k$ and $e_{k+1,1} = 2a_{k+1,1} - 2y'_{k+1}$,

for $x_k \le x \le x_{k+1}$ $k = 1,2,\dots,n-2$, we have from (12)

$$a_{k,1} + a_{k+1,1} = h^{-1}[2(y_{k+1} - y_k) + \frac{h^2}{6}(y''_{k+1} - y''_k) + \frac{h^4}{360}\left(y_k^{(4)} - y_{k+1}^{(4)}\right)],$$

$$\frac{1}{2}\left(2a_{k,1} - 2y'_k + 2y'_k\right) + \frac{1}{2}(2a_{k+1,1} - 2y'_{k+1} + 2y'_{k+1}),$$

$$\frac{1}{2}\left(e_{k,1} + 2y'_k\right) + \frac{1}{2}\left(e_{k+1,1} + 2y'_{k+1}\right) = h^{-1}[2(y_{k+1} - y_k) + \frac{h^2}{6}(y''_{k+1} - y''_k) + \frac{h^4}{360}\left(y_k^{(4)} - y_{k+1}^{(4)}\right)]$$

$$\frac{e_{k,1}+e_{k+1,1}}{2} = h^{-1}[2(y_{k+1} - y_k) - h(y'_{k+1} + y'_i) + \frac{h^2}{6}(y''_{k+1} - y''_k) + \frac{h^4}{360}\left(y_k^{(4)} - y_{k+1}^{(4)}\right)].$$

by using Taylor series expansion for $f(x) \in c^6[0,1]$, about x_k, we have

$$\left|e_{k+1,1}\right| \le \frac{7(k+1)}{360} h^5 w_6(f;h) \to \left|e_{k,1}\right| \le \frac{7k}{360} h^5 w_6(f;h),\ k = 0,1,\dots,n-1$$

Theorem 2: Let $f \in C^6[0,1]$ and $S_\nabla(x) \in S_p(6,3,n)$ be a unique spline function satisfying the condition of Theorem 1. Then

$\left\|s_\nabla^{(l)}(x) - f^{(l)}(x)\right\| \le Kh^{6-l}\, w_6(f;h), l = 0, 1, 2, 3, 4, 5, 6$ and $w_6(f;h)$ denotes the modulus of continuity of $f^{(6)}$.

Where

$$K=\begin{cases}\frac{2407}{720} & \text{, when } x_0 \le x \le x_1 \\ \frac{(1701k+4013)}{720} & \text{, when } x_k \le x \le x_{k+1} \quad ,k=1,2,\dots,n-2 \quad \text{for} \quad l=0,1 \\ \frac{1933}{360} & \text{, when } x_{n-1} \le x \le x_n \end{cases}$$

$$K=\begin{cases}\frac{11}{3} & \text{, when } x_0 \le x \le x_1 \\ \frac{(63k+149)}{24} & \text{, when } x_k \le x \le x_{k+1} \quad ,k=1,2,\dots,n-2 \quad \text{for} \quad l=2,3 \\ \frac{71}{12} & \text{, when } x_{n-1} \le x \le x_n \end{cases}$$

$$K=\begin{cases}\frac{9}{2} & \text{, when } x_0 \le x \le x_1 \\ \frac{(21k+38)}{6} & \text{, when } x_k \le x \le x_{k+1} \quad ,k=1,2,\dots,n-2 \quad \text{for} \quad l=4,5 \\ \frac{15}{2} & \text{, when } x_{n-1} \le x \le x_n \end{cases}$$

$$K=\begin{cases}\frac{8}{3} & \text{, when } x_0 \le x \le x_1 \\ \frac{7(k+3)}{3} & \text{, when } x_k \le x \le x_{k+1} \quad ,k=1,2,\dots,n-2 \quad \text{for} \quad l=6 \\ \frac{14}{3} & \text{, when } x_{n-1} \le x \le x_n \end{cases}$$

Proof: for $x_k \le x \le x_{k+1}$ $k=1,2,\dots,n-2$, we have from (7)

$$\left|s_k^{(6)}(x)-f^{(6)}(x)\right|=\left|720\,a_{k,6}-f^{(6)}(x)\right|$$

by using the Taylor series expansion for $f(x)\in c^6[0,1]$, about x_k, we have:

$$y(x)=y_k+(x-x_k)y_k'+\frac{(x-x_k)^2}{2}y_k''+\frac{(x-x_k)^3}{6}y_k^{(3)}+\frac{(x-x_k)^4}{24}y_k^{(4)}+\frac{(x-x_k)^5}{120}y_k^{(5)}$$
$$+\frac{(x-x_k)^6}{720}y^{(6)}(\xi_k)\,,\, x_k<\xi_k<x\,, \tag{16}$$

$$y'(x)=y_k'+(x-x_k)y_k''+\frac{(x-x_k)^2}{2}y_k^{(3)}+\frac{(x-x_k)^3}{6}y_k^{(4)}+\frac{(x-x_k)^4}{24}y_k^{(5)}$$
$$+\frac{(x-x_k)^5}{120}y^{(6)}(\xi_k)\,,\, x_k<\xi_k<x\,, \tag{17}$$

$$y''(x)=y_k''+(x-x_k)y_k^{(3)}+\frac{(x-x_k)^2}{2}y_k^{(4)}+\frac{(x-x_k)^3}{6}y_k^{(5)}+\frac{(x-x_k)^4}{24}y^{(6)}(\xi_k)\,,$$
$$x_k<\xi_k<x\,, \tag{18}$$

$$y^{(3)}(x)=y_k^{(3)}+(x-x_k)y_k^{(4)}+\frac{(x-x_k)^2}{2}y_k^{(5)}+\frac{(x-x_k)^3}{6}y^{(6)}(\xi_k)\,,\, x_k<\xi_k<x \tag{19}$$

$$y^{(4)}(x)=y_k^{(4)}+(x-x_k)y_k^{(5)}+\frac{(x-x_k)^2}{2}y^{(6)}(\xi_k)\,,\, x_k<\xi_k<x\,, \tag{20}$$

$$y^{(5)}(x)=y_k^{(5)}+(x-x_k)y^{(6)}(\xi_k)\,,\, x_k<\xi_k<x\,, \tag{21}$$

$$y^{(6)}(x) = y^{(6)}(\xi_k)\ ,\ x_k < \xi_k < x\ . \tag{22}$$

Putting $x = x_k$ in (22), using Lemma 1 and (15), we obtain:

$$\left|s_k^{(6)}(x) - f^{(6)}(x)\right| = \left|720\ a_{k,6} - f^{(6)}(x)\right| \leq \frac{7(k+3)}{3}\ w_6(f,h) \tag{23}$$

also since $s_k^{(5)}(x) = 120\ a_{k,5} + 720\ h\ a_{k,6}$, Putting $x = x_k$ in (21), using

Lemma 1, (23), and (14) we get:

$$s_k^{(5)}(x) - f^{(5)}(x) = 120\ a_{k,5} - f^{(5)}(x) + \left(720\ a_{k,6} - f^{(6)}(x)\right).h + hf^{(6)}(x)$$

$$\left|s_k^{(5)}(x) - f^{(5)}(x)\right| \leq h\left|s_k^{(6)}(x) - f^{(6)}(x)\right| + 60\ h^{-4}\left|e_{k+1,1}\right| + \frac{17}{6}h\ w_6(f;h)$$

$$\left|s_k^{(5)}(x) - f^{(5)}(x)\right| \leq \frac{(21k+38)}{6}h\ w_6(f;h) \tag{24}$$

Since $s_k^{(4)}(x) - f^{(4)}(x) = \int_{x_i}^{x}\left(s_k^{(5)}(t) - f^{(5)}(t)\right)dt + s_k^{(4)}(x_i) - f^{(4)}(x_i)$

$s_k^{(4)}(x) - f^{(4)}(x) = \int_{x_i}^{x}\left(s_k^{(5)}(t) - f^{(5)}(t)\right)dt$ by (1) $s_k^{(4)}(x_i) - f^{(4)}(x_i) = 0$

and from (24) we obtain:

$$\left|s_k^{(4)}(x) - f^{(4)}(x)\right| \leq \frac{(21k+38)}{6}h^2\ w_6(f;h)$$

since $s_k^{(3)}(x) = 6\ a_{k,3} + h\ y_k^{(4)} + 60\ h^2 a_{k,5} + 120\ h^3\ a_{k,6}$

$$s_k^{(3)}(x) - f^{(3)}(x) = \frac{h^2}{2}\left(s_k^{(5)}(x) - f^{(5)}(x)\right) - \frac{h^3}{3}\left(s_k^{(6)}(x) - f^{(6)}(x)\right)$$
$$+6\ a_{k,3} + \frac{h^2}{2}f^{(5)}(x) - \frac{h^3}{3}f^{(6)}(x) - h^3 f^{(3)}(x)\ .$$

Putting $x = x_k$ in (19), using Lemma 1, (13), and (23)-(24), we have that:

$$\left|s_k^{(3)}(x) - f^{(3)}(x)\right| \leq \frac{(63k+149)}{24}h^3\ w_6(f;h) \tag{25}$$

$$s''_k(x) - f''(x) = \int_{x_i}^{x}\left(s_k^{(3)}(t) - f^{(3)}(t)\right)dt + s''_k(x_i) - f''(x_i)$$

by (1) $s''_k(x_i) - f''(x_i) = 0$ and from (25) we get:

$$|s''_k(x) - f''(x)| \leq \frac{(63k+149)}{24}h^4\ w_6(f;h)\ .$$

Since $s'_k(x) = a_{k,1} + hy''_k + 3h^2\ a_{k,3} + \frac{h^3}{6}y_k^{(4)} + 5h^4 a_{k,5} + 6h^5 a_{k,6}$

$$s'_k(x_i) - f'(x) = \frac{h^2}{2}\left[s_k^{(3)}(x) - f^{(3)}(x)\right] - \frac{5}{24}h^4\left[s_k^{(5)}(x) - f^{(5)}(x)\right]$$
$$+\frac{2}{15}h^5\left[s_k^{(6)}(x) - f^{(6)}(x)\right] + a_{k,1} + hy''_k + \frac{h^3}{3!}y_k^{(4)}$$
$$+\frac{h^2}{2}f^{(3)}(x) - \frac{5}{24}h^4 f^{(5)}(x) + \frac{2}{15}h^5 f^{(6)}(x) - \frac{h^3}{2}\ y_k^{(4)} - f'(x)\ .$$

Putting $x = x_k$ in (17), by using (12), lemma 1, (23)-(25), we obtain:

$$|s_k'(x) - f'(x)| \le \frac{(1701k+4013)}{720} h^5 w_6(f;h) \tag{26}$$

$s_k(x) - f(x) = \int_{x_i}^{x} \big(s_k'(t) - f'(t)\big)dt + s_k(x_i) - f(x_i)$

by (1) since $s_k(x_i) - f(x_i) = 0$ and using (26), we have that:

$|s_k(x) - f(x)| \le \int_{x_i}^{x} \big(s_k'(t) - f'(t)\big)dt \le \frac{(1701k+4013)}{720} h^6 w_6(f;h)$

$$|s_k(x) - f(x)| \le \frac{(1701k + 4013)}{720} h^6 w_6(f;h)$$

This proves Theorem 2 for $x_k \le x \le x_{k+1}$, $k = 1,2, \dots, n-2$

for $x_0 \le x \le x_1$ we have

from (6) $s_0^{(6)}(x) - f^{(6)}(x) = 720\, a_{0,6} - f^{(6)}(x)$, by using (11), we get:

$$\left|s_0^{(6)}(x) - f^{(6)}(x)\right| \le \frac{8}{3}\, w_6(f;h)\ . \tag{27}$$

Since $s_0^{(5)}(x) = 120\, a_{0,5} + 720\, h\, a_{0,6}$

$\rightarrow s_0^{(5)}(x) - f^{(5)}(x) = 120\, a_{0,5} + 720\, h\, a_{0,6} - f^{(5)}(x)$

By using (27) and (10), we obtain

$$\left|s_0^{(5)}(x) - f^{(5)}(x)\right| \le \frac{9}{2} h\, w_6(f;h) \tag{28}$$

$$s_0^{(4)}(x) - f^{(4)}(x) = \int_{x_i}^{x} \left(s_0^{(5)}(t) - f^{(5)}(t)\right) dt + s_0^{(4)}(x_i) - f^{(4)}(x_i),$$

$s_0^{(4)}(x) - f^{(4)}(x) = \int_{x_i}^{x} \left(s_0^{(5)}(t) - f^{(5)}(t)\right) dt$ by (1) $s_0^{(4)}(x_i) - f^{(4)}(x_i) = 0$

using (28) we have: $\left|s_0^{(4)}(x) - f^{(4)}(x)\right| \le \frac{9}{2} h^2\, w_6(f;h)$ (29)

since $s_0^{(3)}(x) = 6\, a_{0,3} + h\, y_0^{(4)} + 60\, h^2 a_{0,5} + 120\, h^3\, a_{0,6}$

$s_0^{(3)}(x) - f^{(3)}(x) = 6\, a_{0,3} + h\, y_0^{(4)} + 60\, h^2 a_{0,5} + 120\, h^3\, a_{0,6} - f^{(3)}(x)$

$$s_0^{(3)}(x) - f^{(3)}(x) = 6\, a_{0,3} + h\, y_0^{(4)} + \frac{h^2}{2}[s_0^{(5)}(x) - f^{(5)}(x)] - \frac{h^3}{3}[s_0^{(6)}(x) - f^{(6)}(x)]$$
$$-f^{(3)}(x) + \frac{h^2}{2} f^{(5)}(x) - \frac{h^3}{3} f^{(6)}(x)\ .$$

using (9), (27)-(28), we get:

$$\left|s_0^{(3)}(x) - f^{(3)}(x)\right| \le \frac{11}{3} h^3 w_6(f;h) \tag{30}$$

Since $s''_0(x) - f''(x) = \int_{x_i}^{x} \left(s_0^{(3)}(t) - f^{(3)}(t)\right) dt + s_0''(x_i) - f''(x_i)$.

Since by (1) $s_0''(x_i) - f''(x_i) = 0$, using (30), we obtain:

$$|s_0''(x) - f''(x)| \le \frac{11}{3} h^4 w_6(f;h) \tag{31}$$

also $s_0'(x) = y_0' + hy_0'' + 3h^2\, a_{0,3} + \frac{h^3}{6} y_0^{(4)} + 5h^4 a_{0,5} + 6h^5 a_{0,6}$

$$s_0'(x) - f'(x) = y_0' + hy_0'' + 3h^2\, a_{0,3} + \frac{h^3}{6} y_0^{(4)} + 5h^4 a_{0,5} + 6h^5 a_{0,6} - f'(x)$$

$$s_0'(x) - f'(x) = y_0' + hy_0'' + \frac{h^3}{6} y_0^{(4)} + \frac{h^2}{2}[s_0^{(3)}(x) - f^{(3)}(x)] + \frac{h^2}{2} f^{(3)}(x) - f'(x) - \frac{h^3}{2} y_0^{(4)} - \frac{5}{24} h^4 \left[s_0^{(5)}(x) - f^{(5)}(x)\right] + \frac{2}{15} h^5 \left[s_0^{(6)}(x) - f^{(6)}(x)\right]$$

$$s_0'(x) - f'(x) = \frac{h^2}{2}[s_0^{(3)}(x) - f^{(3)}(x)] - \frac{5}{24} h^4 \left[s_0^{(5)}(x) - f^{(5)}(x)\right] + \frac{2}{15} h^5 \left[s_0^{(6)}(x) - f^{(6)}(x)\right] + h^{-1}[hy_0' + h^2 y_0'' + \frac{h^4}{6} y_0^{(4)} + \frac{h^3}{2} f^{(3)}(x) - \frac{h^4}{2} y_0^{(4)} - \frac{5}{24} h^5 f^{(5)}(x) + \frac{2}{15} h^6 f^{(6)}(x) - hf'(x)].$$

Using (30), (27) - (28), we get:

$$|s_0'(x) - f'(x)| \le \frac{2407}{720} h^5 w_6(f; h) \quad (32)$$

since $s_0(x) - f(x) = \int_{x_i}^{x} \big(s_0'(t) - f'(t)\big)\, dt + s_0(x_i) - f(x_i)$

$s_0(x) - f(x) = \int_{x_i}^{x} \big(s_0'(t) - f'(t)\big)\, dt$ by (1), we have that $s_0(x_i) - f(x_i) = 0$

using (32) we obtain: $|s_0(x) - f(x)| \le \frac{2407}{720} h^6\, w_6(f; h)$

for $x_{n-1} \le x \le x_n$, we have

$$s_{n-1}^{(6)}(x) - f^{(6)}(x) = 720\, a_{n-1,6} - f^{(6)}(x) = 240h^{-6}\, [(y_{n-1} - y_n) + hy_n' - \frac{h^2}{6}(2y_n'' + y_{n-1}'') + \frac{h^4}{360}\left(8y_n^{(4)} + 7y_{n-1}^{(4)}\right)] - f^{(6)}(x)\,.$$

Using (16)-(20) for $k = n - 1$ and $x = x_n$, we get:

$$\left|s_{n-1}^{(6)}(x) - f^{(6)}(x)\right| \le \frac{14}{3}\, w_6(f; h)\,. \quad (33)$$

Since $s_{n-1}^{(5)}(x) = 120\, a_{n-1,5} + 720\, h\, a_{n-1,6}$

$$s_{n-1}^{(5)}(x) - f^{(5)}(x) = 120\, a_{n-1,5} - f^{(5)}(x) + 720\, h\, a_{n-1,6}$$

$$s_{n-1}^{(5)}(x) - f^{(5)}(x) = 120\, a_{n-1,5} - f^{(5)}(x) + h\left(720\;\, a_{n-1,6} - f^{(6)}(x)\right) + hf^{(6)}(x),$$

using (33) and (16)-(20) for $k = n - 1$ and $x = x_n$, we get:

$$\left|s_{n-1}^{(5)}(x) - f^{(5)}(x)\right| \le \frac{15}{2} h\, w_6(f; h) \quad (34)$$

$$s_{n-1}^{(4)}(x) - f^{(4)}(x) = \int_{x_i}^{x} \left(s_{n-1}^{(5)}(t) - f^{(5)}(t)\right) dt + s_{n-1}^{(4)}(x_i) - f^{(4)}(x_i)$$

$$s_{n-1}^{(4)}(x) - f^{(4)}(x) = \int_{x_i}^{x} \left(s_{n-1}^{(5)}(t) - f^{(5)}(t)\right) dt\,.$$

Since from (2) $s_{n-1}^{(4)}(x_i) - f^{(4)}(x_i) = 0$ and by using (34), we obtain:

$$\left|s_{n-1}^{(4)}(x)-f^{(4)}(x)\right| \leq \frac{15}{2}h^2\, w_6(f;h)$$

$$s_{n-1}^{(3)}(x)=6\,a_{n-1,3}+h\,y_{n-1}^{(4)}+60\,h^2 a_{n-1,5}+120\,h^3\,a_{n-1,6}$$

$$s_{n-1}^{(3)}(x)-f^{(3)}(x)=6\,a_{n-1,3}+h\,y_{n-1}^{(4)}+60\,h^2 a_{n-1,5}+120\,h^3\,a_{n-1,6}-f^{(3)}(x)$$

$$s_{n-1}^{(3)}(x)-f^{(3)}(x)=h^{-3}[10(y_{n-1}-y_n)+10hy_n'-\frac{1}{3}h^2(7y_n''+8y_{n-1}'')$$
$$+\frac{h^4}{360}\left(20y_n^{(4)}-50y_{n-1}^{(4)}\right)+\frac{h^5}{2}f^{(5)}(x)-\frac{h^6}{3}f^{(6)}(x)]-f^{(3)}(x)$$
$$+\frac{h^2}{2}\left(s_{n-1}^{(5)}(x)-f^{(5)}(x)\right)-\frac{h^3}{3}\left(s_{n-1}^{(6)}(x)-f^{(6)}(x)\right)+h\,y_{n-1}^{(4)}$$

using (33)-(34) and (16)-(22) for $k=n-1$ and $x=x_n$, we get:

$$\left|s_{n-1}^{(3)}(x)-f^{(3)}(x)\right| \leq \frac{71}{12}h^3 w_6(f;h) \tag{35}$$

$$s''_{n-1}(x)-f''(x)=\int_{x_i}^{x}\left(s_{n-1}^{(3)}(t)-f^{(3)}(t)\right)dt+s''_{n-1}(x_i)-f''(x_i)\ .$$

Since from (2) $s''_{n-1}(x_i)-f''(x_i)=0$ and by using (35)

$$|s''_{n-1}(x)-f''(x)| \leq \frac{71}{12}h^4 w_6(f;h)$$

$$s'_{n-1}(x_n)=a_{n-1,1}+hy''_{n-1}+3h^2\,a_{n-1,3}+5h^4 a_{n-1,5}+\frac{h^3}{3!}y_{n-1}^{(4)}+6h^5 a_{n-1,6}$$

$$s'_{n-1}(x_n)-f'(x)=\frac{h^2}{2}\left[s_{n-1}^{(3)}(x)-f^{(3)}(x)\right]-\frac{5}{24}h^4\left[s_{n-1}^{(5)}(x)-f^{(5)}(x)\right]$$
$$+a_{n-1,1}+hy''_{n-1}+\frac{2}{15}h^5\left[s_{n-1}^{(6)}(x)-f^{(6)}(x)\right]+\frac{h^3}{3!}y_{n-1}^{(4)}$$
$$+\frac{h^2}{2}f^{(3)}(x)-\frac{5}{24}h^4 f^{(5)}(x)+\frac{2}{15}h^5 f^{(6)}(x)-\frac{h^3}{2}\ y_{n-1}^{(4)}-f'(x)$$

$$s'_{n-1}(x_n)-f'(x)=\frac{h^2}{2}\left[s_{n-1}^{(3)}(x)-f^{(3)}(x)\right]-\frac{5}{24}h^4\left[s_{n-1}^{(5)}(x)-f^{(5)}(x)\right]$$
$$+\frac{2}{15}h^5\left[s_{n-1}^{(6)}(x)-f^{(6)}(x)\right]+h^{-1}[2(y_n-y_{n-1})-hy_n'$$
$$+\frac{h^2}{6}(y_n''-y_{n-1}'')-\frac{h^4}{360}\left(y_n^{(4)}-y_{n-1}^{(4)}\right)+h^2 y''_{n-1}-\frac{2h^4}{3!}y_{n-1}^{(4)}$$
$$+\frac{h^3}{2}f^{(3)}(x)-\frac{5}{24}h^5 f^{(5)}(x)+\frac{2}{15}h^6 f^{(6)}(x)-hf'(x)]$$

using (33)-(35) and (16)-(22) for $k=n-1$ and $x=x_n$, we get:

$$|s'_{n-1}(x_n)-f'(x)| \leq \frac{1933}{360}h^5 w_6(f;h) \tag{36}$$

$$s_{n-1}(x)-f(x)=\int_{x_i}^{x}\left(s'_{n-1}(t)-f'(t)\right)dt+s_{n-1}(x_i)-f(x_i)$$

since from (2) $s_{n-1}(x_i)-f(x_i)=0$, using (36), we obtain:

$$|s_{n-1}(x)-f(x)| \leq \int_{x_i}^{x}\left(s'_{n-1}(t)-f'(t)\right)dt=\frac{1933}{360}h^6 w_6(f;h)$$

$|s_{n-1}(x) - f(x)| \leq \frac{1933}{360} h^6 w_6(f;h).$

This completes the proof of Theorem 2.

CONCLUSION

In this paper, we have devised an approximation approach for calculating the error bound using the lacunary spline polynomial functions. The type of lacunary data does not affect the pace of convergence, but the end condition needs to be carefully considered.

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References

[1] R. G. Karem, K. H. F. Jwamer, and F. K. Hamasalh, "Lacunary interpolation by the spline function of fractional degree," *Journal of Al-Qadisiyah for Computer Science and Mathematics*, vol. 15, no. 3, pp. 54–72, 2023. DOI: 10.29304/jqcm.2023.15.3.1282.

[2] K. H. Jwamer, "Study some of the lacunary interpolation by splines and their error bounds," M.Sc. Thesis, Salahaddin University, Erbil, Iraq, 2001.

[3] K. H. Jwamer, "Minimizing error bounds in (0, 2, 3) lacunary interpolation by sixtic spline," *American Journal of Mathematics and Statistics*, vol. 3, no. 4, pp. 249–256, 2007.

[4] R. G. Karem, K. H. F. Jwamer, and F. K. Hamasalh, "Existence, uniqueness, and stability results for fractional differential equations with lacunary interpolation by the spline method," *Mathematics and Statistics*, vol. 11, no. 4, pp. 669–675, 2023. DOI: 10.13189/ms.2023.110408.

[5] F. K. Hamasalh and K. H. F. Jwamer, "On optimality of lacunary interpolation for recovery of C6C^6C6 seventh-degree spline," *International Journal of Open Problems in Computer Mathematics*, vol. 5, no. 1, Mar. 2012. DOI: 10.12816/0006097.

[6] K. H.-F., F. K. H.-S., and R. K. G., "Approximation new error bounded by spline degree six," *Acta Universitatis Apulensis*, Special Issue, Romania, 2011.

[7] K. H. Jwamer and F. K. Hama-Salh, "Minimizing error bounds in lacunary interpolation by spline (0, 2) case," *Journal of Kirkuk University - Scientific Studies*, vol. 4, no. 2, pp. 117–124, 2009.

[8] S. R. K. and K. H. F. Jwamer, "Lacunary interpolation by spline function - (0, 1, 4) case," *Journal of Dohuk University*, vol. 4, no. 2, pp. 193–196, 2001. Available: ResearchGate.

[9] G. M. Phillips, *Interpolation and Approximation by Polynomials*. New York: Springer Verlag, 2003.

[10] R. K. Saeed and K. H. Jwamer, "Minimizing error bounds in lacunary interpolation by spline function (0, 1, 4) case," *Journal of Al-Nahrain University*, vol. 8, no. 2, pp. 114–119, 2005.

[11] R. K. Saeed, "(0, 1, 3, 5) Lacunary interpolation by deficient splines," *Journal of Science and Nature*, vol. 2, no. 1, pp. 28–30, 1993. DOI: 10.12691/ajna-4-1-3.

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ركز الدراسة على الاندماج فراغية وتقديرات الخطأ باستخدام دالة السبلاين

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الملخص

من أجل حل مشكلة الاندراج فراغية بمدى (0، 2، 4) فإننا ننظر أولاً إلى فئة مناسبة من االسبلاينات غير الكافية. وفي ظل الافتراضات المناسبة، نثبت وجود الحل وتفرده. نحن نصف تحليل التقارب ونبين أن النهج هو من الترتيب السادس، وأنه يمكن بناؤه من أي ترتيب، مع أي متطلبات الاندراج وسيطة.

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