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# Reproducing Kernel Hilbert Space Method for Solving Nonlinear Integro-Differential Equations

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A B S T R A C T

In this work, the reproducing kernel Hilbert space method (RKHSM) was used to find a numerical solution to nonlinear integro-differential equations (NIDEs) on the form of finite series. The results showed that the approximate solution are converges to the exact solution of the NIDEs, which confirms the effectiveness of RKHSM as a reliable and practical tool for solving this type of e. This method is of great importance in the field of numerical solutions of NIDEs, which has wide applications in many engineering and scientific fields

## 1. Introduction

Integro-differential equations (IDEs) are important in describing physical, natural, and biological phenomena, as well as in engineering and biological technologies, in recent years, they have received increasing attention in many different disciplines. Such as elasticity theory, biomechanics, electromagnetism, industry and scientific applications, and others [1, 2, 3].

RKHSM considered one of the effective methods to solve many papers, including differential and integral equations which have widely used in statistics and probability.

The following formula of NIDEs is given by [4]:

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$$\sum_{k=1}^{q} b_k(t) g^{(k)}(t) = w(t) + \sigma \int_{t_0}^{t_1 or\, t} T(t,z) F(g(z)) dz\,, \tag{1}$$

the initial condition $g^{(k)}(t_0) = r_k \quad k = 1, \dots, q,$
where $b_k(t),\ w(t)$ and $T(t,z)$ are known functions so that $t \in [t_0, t_1], t_0, t_1, \sigma, r_k, q \in \mathbb{R}$ , $F(g(t))$ is a nonlinear operator and $g(t)$ is an unknown function in $W_2^i[\,t_0, t_1], i = 1,2, \dots$ .
IDEs have wide applications in various fields, during the last decades, many numerical methods have been developed to solve these equations, whether linear or nonlinear. These methods include Adomian decomposition method by [5] collection method [6, 7, 8, 9], Taylor polynomial [10], cubic B-spline least-square method combine with a quadratic weight function [11], cubic B-spline Galerkin method combine with a quadratic weight function[12], least-squares method [13, 14], homotopy perturbation method [15], the simplified reproducing kernel method and the homotopy perturbation method [16], Laplace discrete Adomian decomposition method [17] , other numerical methods are used to solve IDEs , since these equations are difficult to solve using analytical methods.
The RKHSM has been used to solving many problems, such as singular IEs [18, 19, 20, 21], linear IEs [22, 23, 24], nonlinear IEs [25, 26, 27] and all equations were solved using the Gram-Schmidt algorithm. However, the researchers below worked on a new approach, where they solved a nonlinear IEs [28], solved a class of functional IEs [29] and solved third-order differential equations [30].
The main purpose of this paper is to apply RKHSM for solving NIDEs. The convergence and error analysis will be discussed.
It is organized into five parts, including the introduction. The second part gives the basic definitions of the reproducing kernel, inner product, norm, function space, and the form of the reproducing kernel. The third part explains how to apply the method to NIDEs, and also discusses the convergence theory. The fourth part gives the numerical results and explains the error magnitude, and the fifth part contains the conclusion.

## 2. Reproducing Kernel Hilbert Space

In this part, It has been defined the function space $W_2^i[t_0, t_1]$ where $i = 1,2, \dots$ and define the kernel $G_s(t)$ in any space in the research that relies on $W_2^i[t_0, t_1]$, [31]:

**Definition 2.1.** Let it be $S$ are abstract set so that $S \neq \phi$, a function $G: S \times S \to \mathbb{C}$, where $\mathbb{C}$ be the set of complex numbers $G$ is a reproducing kernel of H if satisfied

- $\forall s \in S, G_s(t) \in H,$
- $\forall s \in S, \forall g \in H, g(s) = < G_s(t), g(t) >.$ (2)

**Definition 2.2.** The definition of function space $W_2^i[t_0, t_1], i = 1,2, \dots$ is as follows:

$$W_2^i[t_0, t_1] = \{g(t): g: [t_0, t_1] \to \mathbb{R}, g^{(i-1)} continuous\ function, g^{(i)} \in L^2[t_0, t_1]\},$$

The inner product and the norm in $W_2^i[t_0, t_1]$ are defined as follows.

$$< g, f >_{W_2^i[t_0,t_1]} = \sum_{k=0}^{i-1} g^{(k)}(t_0) f^{(k)}(t_0) + \int_{t_0}^{t_1} g^{(i)}(t) f^{(i)}(t) dt, \tag{3}$$

$$\|g\|_{W_2^i[t_0,t_1]} = \sqrt{< g, g >_{W_2^i[t_0,t_1]}} \tag{4}$$

**Remark 2.1.** [25] If $\forall\, t \in [t_0, t_1]$ and $\forall g(t) \in W_2^i[t_0, t_1]\ \exists\ G_s(t) \in W_2^i[t_0, t_1]$ such that $< G_s(t), g(s) >_{W_2^i[t_0,t_1]} = g(t)$, where $s \in [t_0, t_1]$ then the Hilbert space $W_2^i[t_0, t_1]$ is a reproducing kernel space.

Define the reproducing kernel of $W_2^i[t_0, t_1]$ is

$$G_s(t)=\begin{cases} G1_s(t)=\sum_{k=1}^{2i} a_k(s)t^{k-1},\ t\le s\\ G2_s(t)=\sum_{k=1}^{2i} v_k(s)t^{k-1},\ t>s\end{cases} \tag{5}$$

the coefficients $a_k(s), v_k(s),\ \{k=1,2,\dots,2i\}$, can be obtained by.

$$\frac{\partial^k G1_s(s)}{\partial t^k}=\frac{\partial^k G2_s(s)}{\partial t^k},\ k=0,1,\dots,2i-2, \tag{6}$$

and

$$(-1)^i\left(\frac{\partial^{2i-1}G1_s(s^+)}{\partial t^{2i-1}}-\frac{\partial^{2i-1}G2_s(s^-)}{\partial t^{2i-1}}\right)=1. \tag{7}$$

**Remark 2.2.** By using (6) and (7), we will calculate $G_s(t)$ in $W_2^i[0,1]$ when $i=1,2,3,4$

$i=4$,

$$G_s^4(t)=\begin{cases}\frac{-t^7}{5040}+\frac{(t^6s)}{720}-\frac{(t^5s^2)}{240}+\frac{(t^4s^3)}{144}+\frac{(t^3s^3)}{36}+\frac{(t^2s^2)}{4}+ts+1, & t\le s\\ \frac{-s^7}{5040}+\frac{(s^6t)}{720}-\frac{(s^5t^2)}{240}+\frac{(s^4t^3)}{144}+\frac{(s^3t^3)}{36}+\frac{(s^2t^2)}{4}+ts+1, & t>s\end{cases}$$

$i=3$,

$$G_s^3(t)=\begin{cases}\frac{t^5}{120}-\frac{(t^4s)}{24}+\frac{(t^3s^2)}{12}+\frac{(t^2s^2)}{4}+st+1, & t\le s\\ \frac{s^5}{120}-\frac{(s^4t)}{24}+\frac{(s^3t^2)}{12}+\frac{(s^2t^2)}{4}+st+1, & t>s\end{cases}$$

$i=2$ ,

$$G_s^2(t)=\begin{cases}-\frac{t^3}{6}+\frac{(t^2s)}{2}+st+1, & t\le s\\ -\frac{s^3}{6}+\frac{(s^2t)}{2}+st+1, & t>s\end{cases}$$

$i=1$,

$$G_s^1(t)=\begin{cases}t+1, & t\le s\\ s+1, & t>s\end{cases}$$

## 3. The Implementation of Method

In this part, it is mentioned how RKHSM is applied to solve NIDEs and related theories are mentioned. In many previous studies, researchers used the Gram-Schmidt algorithm to obtain an orthogonal system. However, there are some problems associated with using this algorithm, namely numerical instability due to the large number of computational steps required to implement it. Therefore, in this study, an alternative algorithm that is numerically stable and less computationally complicated will be used.

The equation (1) can be expressed in the following form:

$$Lg(t)=\sum_{k=1}^{q} b_k(t)g^{(k)}(t)-\sigma\int_{t_0}^{t_1 or\ t} T(t,z)F(g(z))dz \tag{8}$$

$Lg(t)=w(\text{t})$,

where $L: W_2^i[t_0,t_1]\to W_2^1[t_0,t_1]$ is an invertible bounded linear operator.

Was chosen a countable set of points $\{t_k\}_{k=1}^{\infty}$ in the interval $[t_0,t_1]$ define

$\delta_k(t)=G_{t_k}^1(t),\quad \mu_k(t)=L^*\delta_k(t)$,

where $L^*$ is the adjoint operator of $L$.

**Theorem 3.1.** [31] Let $\{t_k\}_{k=1}^{\infty}$ is dense on $[t_0,t_1]$ then $\{\mu_k(t)\}_{k=1}^{\infty}$ is a complete function system of $W_2^i[t_0,t_1]$ and $\mu_k(t)=L_sG_{t_k}^i(t)$, where the operator $L_s$ applies to function of $s$

**Theorem 3.2.** [28, 29, 30] Let $\{\mu_k(t)\}_{k=1}^{\infty}$ be a linear independent set in $W_2^i[t_0,t_1]$, and $\{\tilde{\mu}_k(t)\}_{k=1}^{\infty}$ be an orthogonal system in $W_2^i[t_0,t_1]$, so that $\tilde{\mu}_k(t)=\sum_{j=1}^{k}\beta_{kj}\mu_j(t)$, if

$$g(t)=\sum_{k=1}^{\infty}A_k\mu_k(t) \tag{9}$$

and the approximate solution

$$g_M(t)=\sum_{k=1}^{M}A_k\mu_k(t)=\sum_{k=1}^{M}\tilde{A}_k\tilde{\mu}_k(t) \tag{10}$$

then $\Gamma A=\bar{w}$,
where,

$$\Gamma=\begin{pmatrix} <L\mu_1,\mu_1>_{W_2^i[t_0,t_1]} & \cdots & L\mu_1,\mu_M>_{W_2^i[t_0,t_1]} \\ \vdots & \ddots & \vdots \\ <L\mu_M,\mu_1>_{W_2^i[t_0,t_1]} & \cdots & <L\mu_M,\mu_M>_{W_2^i[t_0,t_1]}\end{pmatrix},\ A=\begin{pmatrix}A_1\\ \vdots\\ A_M\end{pmatrix}, \check{A}=\begin{pmatrix}\tilde{A}_1\\ \vdots\\ \tilde{A}_M\end{pmatrix},$$

$\bar{w}=\begin{pmatrix}<w,\mu_1>_{W_2^i[t_0,t_1]}\\ \vdots \\ <w,\mu_M>_{W_2^i[t_0,t_1]}\end{pmatrix},\ \beta=\begin{pmatrix}\beta_{11} & \cdots & 0\\ \vdots & \ddots & \vdots\\ \beta_{1M} & \cdots & \beta_{MM}\end{pmatrix}$, and $\beta_{jh}$ are orthogonal coefficients, $(\beta_{kk}>0, k,h=1,2,\dots,M)$.

**Proof.** Let $g(t)\in W_2^i[t_0,t_1]$ then $g(t)=\sum_{k=1}^{\infty}A_k\mu_k(t)=\sum_{k=1}^{\infty}\tilde{A}_k\tilde{\mu}_k(t)$
To get an approximate solution, was cut the series at $M$.

$$g_M(t)=\sum_{k=1}^{M}A_k\mu_k(t)=\sum_{k=1}^{M}\tilde{A}_k\tilde{\mu}_k(t)$$
$$=\sum_{k=1}^{M}\tilde{A}_k\sum_{j=1}^{k}\beta_{kj}\mu_j(t)=\sum_{j=1}^{M}(\sum_{k=j}^{M}\tilde{A}_k\beta_{kj})\mu_j(t)$$

$\{\mu_k(t)\}_{k=1}^{\infty}$ be a linear independence, the $A_j=\sum_{k=j}^{M}\tilde{A}_k\beta_{kj}, k=1,\dots,M$ then

$$A=\beta^T\tilde{A}, \tag{11}$$

using (8) leads to $Lg_M(t)=w(t)$, for $k=1,\dots,M$ ,

$$\rightarrow\sum_{h=1}^{M}\tilde{A}_h<L\tilde{\mu}_h,\tilde{\mu}_k>_{W_2^i[t_0,t_1]}=<w,\tilde{\mu}_k>_{W_2^i[t_0,t_1]}$$
$$\rightarrow\sum_{h=1}^{M}\tilde{A}_h\sum_{j=1}^{k}\beta_{kj}\sum_{l=1}^{h}\beta_{hl}<L\mu_l,\mu_j>_{W_2^i[t_0,t_1]}=\sum_{j=1}^{k}\beta_{kj}<w,\mu_j>_{W_2^i[t_0,t_1]}$$
$$\rightarrow\sum_{h=1}^{M}\tilde{A}_h\sum_{j=1}^{j}\sum_{l=1}^{h}\beta_{kj}<L\mu_l,\mu_j>_{W_2^i[t_0,t_1]}\beta_{lh}^T=\sum_{j=1}^{k}\beta_{kj}<w,\mu_j>_{W_2^i[t_0,t_1]}$$
$$\rightarrow\sum_{h=1}^{M}\tilde{A}_h(\beta\Gamma\beta^T)_{kh}=\sum_{j=1}^{j}\beta_{kj}<w,\mu_j>_{W_2^i[t_0,t_1]}\rightarrow(\beta\Gamma\beta^T)\tilde{A}=\beta\bar{w},$$

from (11) then $\Gamma A=\bar{w}$. ■

**Theorem 3.3.** [25, 28] If $g(t)\in W_2^i[t_0,t_1]$, then $g(t)$ is bounded.

**Theorem 3.4.** [25, 26, 28, 31] Let $\|g_M(t)\|_{W_2^i[t_0,t_1]}$, is bounded, and Eq. (1) has a unique solution. If the dense sequence $\{t_k\}_{k=1}^{\infty}$,on $[t_0,t_1]$, then M-term approximate solution $g_M(t)$ converges to g$(t)$ of (1) and the exact solution is expressed $g(t)=\sum_{k=1}^{\infty}A_k\mu_k(t)$,where $A_k=\sum_{j=1}^{k}\Gamma_{kj}^{-1}\bar{w}_j$.

**Proof.**

**i.** To prove that $g_M(t)$ convergent to $g(t)$, we conclude that:

$$g_{M+1}(t) = g_M(t)+A_{M+1}\mu_{M+1}(t)=g_M(t)+\tilde{A}_{M+1}\tilde{\mu}_{M+1}(t),$$

$\{\tilde{\mu}_k(t)\}_{k=1}^{\infty}$is orthogonal, and leads to $\|g_{M+1}\|_{W_2^i[t_0,t_1]} = \|g_M\|_{W_2^i[t_0,t_1]}+\tilde{A}_k^2 = \sum_{k=1}^{M}\tilde{A}_k^2$ ,

from the boundedness of $\|g_M\|_{W_2^i[t_0,t_1]}$,$\sum_{k=1}^{M}\tilde{A}_k^2 < \infty$ ,i.e. $\left\{\tilde{A}_k^2\right\}_{k=1}^{\infty} \in J^2$ , where

$J^2 = \left\{\tilde{A}_k; \sum_{k=1}^{M}\tilde{A}_k^{\ 2} < \infty\right\}$, $(k = 1,2,\dots)$ if $N > M$ then

$$\begin{aligned}
&\|g_N - g_M\|^2_{W_2^i[t_0,t_1]} \\
&= \|g_N - g_{N-1} + g_{N-1} - \cdots + g_{M+1} - g_M\|^2_{W_2^i[t_0,t_1]} \\
&\le \|g_N - g_{N-1}\|^2_{W_2^i[t_0,t_1]} + \cdots + \|g_{M+1} - g_M\|^2_{W_2^i[t_0,t_1]} \\
&=\sum_{k=N}^{M+1}\|g_k - g_{k-1}\|^2_{W_2^i[t_0,t_1]}
\end{aligned}$$

So $\|g_k - g_{k-1}\|^2_{W_2^i[t_0,t_1]} = \tilde{A}_k^{\ 2}$, consequently $\|g_N - g_M\|^2_{W_2^i[t_0,t_1]} = \sum_{k=1}^{M}\tilde{A}_k^{\ 2} \to 0$ as $M \to \infty$ . Considering the completeness of $W_2^i[t_0,t_1]$ ,

$$g_M(t) \to g(t), as\ M \to \infty$$

where $g(t) \in W_2^i[t_0,t_1]$.

**ii.** Can prove that $g(t)$ is a solution of (8). By (9)

$g(t) = \sum_{k=1}^{\infty}A_k\mu_k(t) = \sum_{k=1}^{\infty}\tilde{A}_k\tilde{\mu}_k(t)$ ,

Since $Lg(t_j) = \sum_{k=1}^{\infty}\tilde{A}_k < L\tilde{\mu}_k, \delta_j >_{W_2^1[t_0,t_1]}$

$$= \sum_{k=1}^{\infty}\tilde{A}_k < \tilde{\mu}_k, L^*\delta_j >_{W_2^i[t_0,t_1]} = \sum_{k=1}^{\infty}\tilde{A}_k < \tilde{\mu}_k, \mu_j >_{W_2^i[t_0,t_1]},$$

$\to \sum_{j=1}^{M}\beta_{Mj} * (Lg(t_j) = \sum_{k=1}^{\infty}\tilde{A}_k < \tilde{\mu}_k, \mu_j >_{W_2^i[t_0,t_1]})$

$\to \sum_{j=1}^{M}\beta_{Mj}Lg(t_j) = \sum_{k=1}^{\infty}\tilde{A}_k < \tilde{\mu}_k, \sum_{j=1}^{M}\beta_{Mj}\ \tilde{\mu}_j >_{W_2^i[t_0,t_1]}$

$= \sum_{k=1}^{\infty}\tilde{A}_k < \tilde{\mu}_k, \tilde{\mu}_M >_{W_2^i[t_0,t_1]} = \tilde{A}_M = \sum_{j=1}^{M}\beta_{Mj}w(t_j)$.

if $M = 1$ then $\beta_{11}L\ g(t_1)$=$\beta_{11}w(t_0) \to Lg(t_1)$ =$w(t_1)$,

if $M = 2$ then $\beta_{21}L\ g(t_1)$+ $\beta_{22}L\ g(t_2)$= $\beta_{21}w(t_0) + \beta_{22}\ w(t_2) \to Lg(t_2)$ =$w(t_2)$,

moreover, it is clear $Lg(t_j)$ =$w(t_j)$,

Since $\{t_k\}_{k=1}^{\infty}$,is dense on $[t_0,t_1]$, $\forall\ t \in [t_0,t_1]$, $\exists\ \{t_{nk}\}_{k=1}^{\infty}$such that $t_{nk} \to t, as\ k \to \infty$.
Hence, when $k \to \infty$, by $Lg(t)$ =$w(t_k)$, Obtained $g(t)$ is the solution of (8). ∎

**Theorem 3.5.** [25, 26] If $g(t) \in W_2^i[t_0,t_1]$ then $e_M^2 = \|g_M(t) - g(t)\|^2_{W_2^i[t_0,t_1]}$ convergent to 0, and the sequence $\{e_M\}$ is monotonically decreasing .

**Proof.** From $g_M(t), g(t)\ in$ (9) , (10) then

$$\begin{aligned}
e_M^2 = \|g_M - g\|^2_{W_2^i[a,b]} &= \|\textstyle\sum_{k=M+1}^{\infty}A_k\mu_k(t)\|^2_{W_2^i[t_0,t_1]} \\
&= \left\|\textstyle\sum_{k=M+1}^{\infty}\tilde{A}_k\tilde{\mu}_k(t)\right\|^2_{W_2^i[t_0,t_1]} = \textstyle\sum_{k=M+1}^{\infty}\tilde{A}_j^{\ 2},
\end{aligned}$$

and $e_{M-1}^2 = \sum_{k=M}^{\infty}\tilde{A}_k^{\ 2}$ , clearly, $e_M^2 \le e_{M-1}^2$, ,and consequently $\{e_M\}$, is monotonically decreasing. ∎

**Algorithm**

1. $t \in [t_0, t_1]$.
2. $t_k = \frac{(k-1)}{(M-1)}, k = 1,2, \dots, M$.
3. $\mu_k(t) = L_s G^i_{t_k}(t)$.
4. $\Gamma_{kj} =< L\mu_j(t), \mu_k(t) >_{W^i_2[t_0,t_1]}$.
5. N=1.
6. $g_0(t_1) = g(t_1)$.
7. $A_N = \sum_{j=1}^{N} \Gamma^{-1}_{Nj} \overline{w}_j$.
8. $g_N(t) = \sum_{k=1}^{N} A_k \mu_k(t)$.
9. $If\ N = M$ stop.
10. Else $N = N + 1$ go to (7) .

## Numerical experiment

In this study, three examples of NIDEs have been solved to demonstrate the effectiveness of the RKHSM and its ability to converge toward the exact solution after a few number of iterations. Our error rate was also compared with other methods using absolute error as a measure.

The absolute error:

$$e_k = |g(t_k) - g_M(t_k)|, \qquad k = 1,2, \dots, M.$$

**Example 1:** Consider the Volterra NIDE [32]:

$$g'(t) + g(t) = w(t) + 2\int_0^t \sin(t)\, g^2(z)dz, \qquad t \in [0,1] \qquad (12)$$

$where\ w(t) = cos(t)\left(1 + sin^2(t)\right) + (1 - t)sin(t)$, subject to initial condition $g(0) = 0$ and the exact solution is $g(t) = sin(t)$.

**Table 1.** Comparing the numerical results of our method with [32]

| $t$ | $g(t)$ | $g_M(t)$ in $W_2^4$ | $e_k$ | $e_k$ in [32] |
|---|---|---|---|---|
| $2.0000E-01$ | $1.9866E-01$ | $1.9866E-01$ | 0 | $6.6518E-06$ |
| $4.0000E-01$ | $3.8941E-01$ | $0.3894E-01$ | $5.5622E-14$ | $2.2069E-05$ |
| $6.0000E-01$ | $5.6464E-01$ | $5.6464E-01$ | 0 | $4.0620E-05$ |
| $8.0000E-01$ | $7.1735E-01$ | $7.1735E-01$ | 0 | $2.5674E-05$ |
| $1.0000E-00$ | $8.4147E-01$ | $8.4147E-01$ | 0 | $9.6104E-05$ |

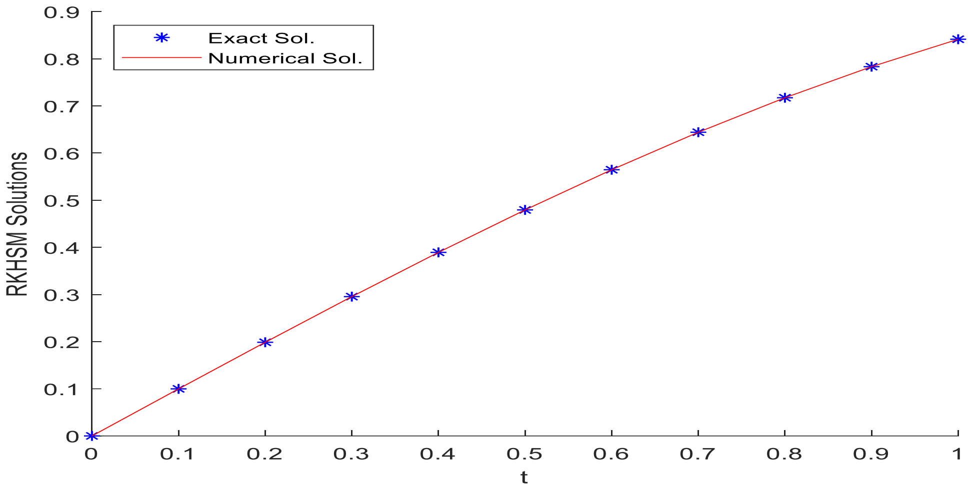


**Fig. 1** The comparisons between approximate and exact solutions for Example 1. at M=6.

**Example 2:** Consider the Fredholm NIDE [9]:

$$g'(t) + g(t) = w(t) - \int_0^1 g^2(z)dz \ , \qquad t \in [0,1] \qquad (13)$$

where $w(t) = \frac{1}{2}(e^{-2} - 1)$, subject to initial condition $g(0) = 1$, and the exact solution is $g(t) = e^{-t}$.

**Table 2.** Comparing the numerical results of our method with [9]

| $t$ | $g(t)$ | $g_M(t)$ in $W_2^5$ | $e_k$ | $e_k$ in [9] |
|---|---|---|---|---|
| $1.2500E-01$ | $8.8249E-01$ | $8.8249E-01$ | $0$ | $6.2983E-13$ |
| $2.5000E-01$ | $7.7880E-01$ | $7.7880E-01$ | $1.1657E-14$ | $6.4948E-13$ |
| $3.7500E-01$ | $6.8728E-01$ | $6.8728E-01$ | $0$ | $6.7090E-13$ |
| $5.0000E-01$ | $6.0653E-01$ | $6.0653E-01$ | $0$ | $6.9033E-13$ |
| $6.2500E-01$ | $5.3526E-01$ | $5.3526E-01$ | $0$ | $7.0754E-13$ |
| $7.5000E-01$ | $4.7236E-01$ | $4.7236E-01$ | $5.8287E-15$ | $7.2214E-13$ |
| $8.7500E-01$ | $4.1686E-01$ | $4.1686E-01$ | $0$ | $7.3024E-13$ |

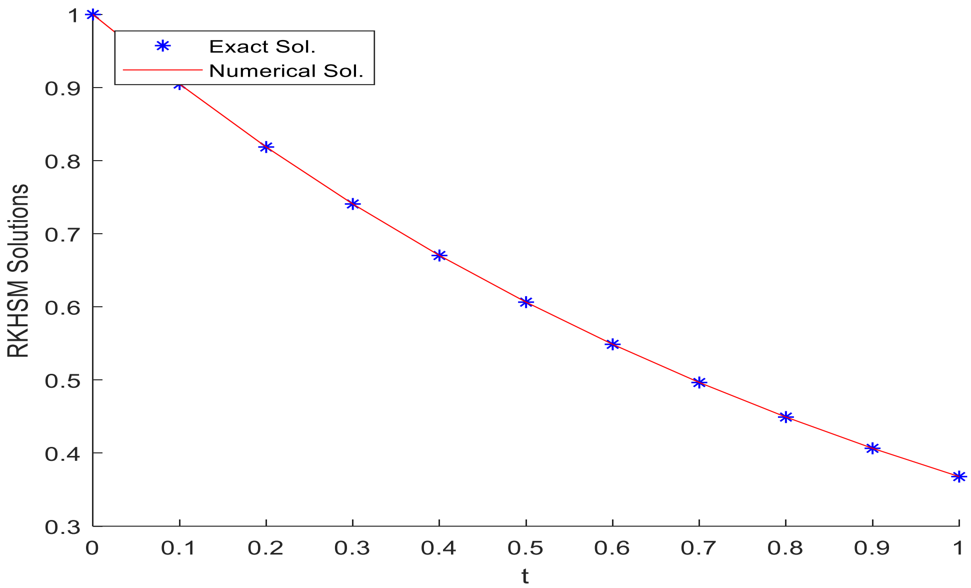

**Fig. 2** The comparisons between approximate and exact solution at M=10.

**Example 3:** Consider the Volterra NIDE [33]:

$$g'(t) = w(t) - \int_0^t (g^2(z) - 2)dz \qquad t \in [0,1] \qquad (14)$$

where $w(t) = \frac{1}{5}t^5$, the initial condition $g(0) = 0$, the exact solution is $g(t) = t^2$

**Table 3.** Comparing the numerical results of our method with [21]

| $t$ | $g(t)$ | $g_M(t)$ in $W_2^6$ | $e_k$ | $e_k$ in [21] |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $2.0000E-01$ | $0.4000E-01$ | $0.4000E-01$ | 0 | $4.6837E-17$ |
| $4.0000E-01$ | $1.6000E-01$ | $1.6000E-01$ | 0 | $2.7755E-17$ |
| $6.0000E-01$ | $3.6000E-01$ | $3.6000E-01$ | $1.1657e-14$ | $1.11022E-16$ |
| $8.0000E-01$ | $6.4000E-01$ | $6.4000E-01$ | 0 | $2.2204E-16$ |
| $1.0000E-00$ | $1.000E-00$ | $1.000E-00$ | 0 | $1.1102E-15$ |

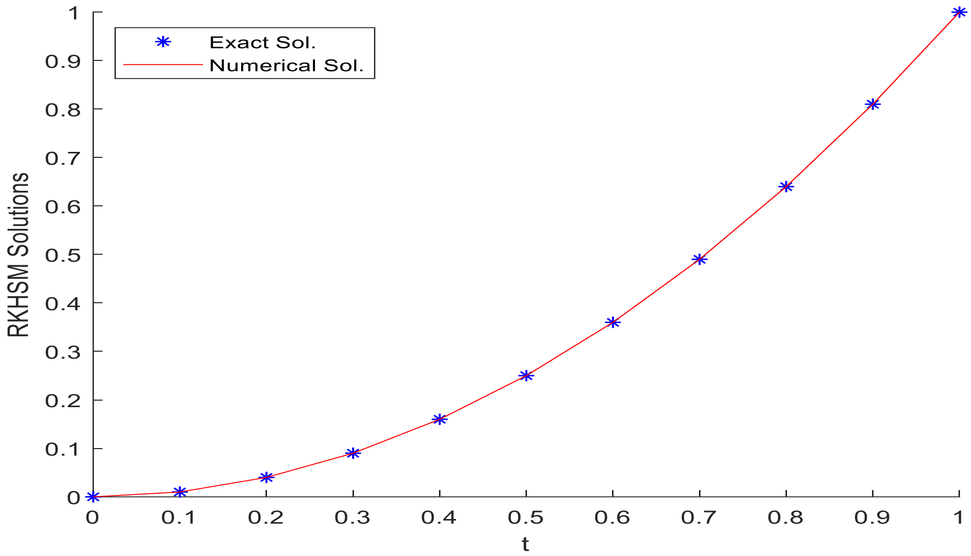


**Fig. 3** The comparisons between approximate and exact solutions at M=4.

## 5. Conclusion

In this study, the reproducing kernel Hilbert space method was applied on three examples of nonlinear integro- differential equations that have analytical solutions for the purpose of knowing the accuracy and effectiveness of proposed method, using MATLAB R2023a to obtain a numerical result. Anew algorithm was used instead of the Gram-Schmidt algorithm, which was It is numerically unstable and contains many complicated mathematical operations. A comparison was also made between the resulting approximate solutions and the approximate solutions of other methods. The results showed that the absolute error in the solutions is equal to zero or converges to zero, which means that it is better than the results of [32], [9] and |33] at an equal number of iterations. The algorithm is simpler, more efficient and contains fewer mathematical steps, leading to more accurate approximate solutions compared to other methods.

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# طريقة استنساخ نواة فضاء هيلبرت لحل المعادلات التفاضلية التكاملية غير الخطية

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**معلومات البحث**

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**الكلمات المفتاحية**

طريقة استنساخ نواة الفضاء هيلبرت ، المعادلات التفاضلية التكاملية غير الخطية ، الحل التقريبي ، فضاء هيلبرت ، فضاء الدوال.

**الملخص**

في هذا العمل تم استخدام طريقة استنساخ نواة فضاء هيلبرت لايجاد حلول التقريبية للمعادلات التفاضلية التكاملية غير الخطية على شكل متسلسلة منتهية، وقد أظهرت النتائج أن الحلول التقريبي يقترب بشكل وثيق من الحلول الدقيق للمعادلات، مما يؤكد فعالية الطريقة كأداة موثوقة وعملية لحل هذا النوع من المعادلات. هذه الطريقة ذات أهمية كبيرة في مجال ايجاد الحلول العددية للمعادلات التفاضلية التكاملية غير الخطية، والتي لها تطبيقات واسعة في العديد من المجالات الهندسية والعلمية.

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