



# The Effect of Convolution Theory for Heat Transfer of Unsteady Nanofluid Flow with Presenting an Inclined Magnetic Field

Haedir Abd AlRazak Namooos\* , Abeer Majeed Jasim 

Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq.

## ARTICLE INFO

Received 5 January 2025  
Revised 7 February 2025  
Accepted 15 February 2025  
Published 31 June 2025

## Keywords :

Q-Homotopy Analysis Method, Laplace Transform, Convolution Theory, Padé Approximate, Variable Thermal Conductivity, Squeezing Unsteady, An Inclination Angle Of An Applied Magnetic Field.

## ABSTRACT

The study examines unsteady nanofluid heat transfer in a squeezing flow between two parallel plates, using water as the base of fluid with gold ( $Au$ ), and magnetite ( $Fe_3O_4$ ) nanoparticles. A new analytical approach (LCP-q-HAM) combining the q-homotopy analysis method, Laplace transform, convolution theory, and Padé approximation is employed to solve nonlinear differential equations that have to do with magnetic field effects and thermal conductivity. Analytical and numerical solutions (using BVP4C) are compared through tables and graphs, analyzing temperature and velocity distributions for various parameters such as nanoparticle volume fraction, Hartmann number, squeeze number, magnetic field angle, and conductivity. The results confirm the effectiveness of the analytical method.

**Citation:** H. A. A. Namooos, A. M. Jasim, J. Basrah Res. (Sci.) 51(1), 68 (2025).  
[DOI:https://doi.org/10.56714/bjrs.51.1.6](https://doi.org/10.56714/bjrs.51.1.6)

## 1. Introduction

Heat transfer squeezing unsteady viscous flow between moving plates has gained considerable interest from scientists due to its wide range in many physical applications, including lubrication systems, polymer processing, food processing, hydrodynamical machines, compression, and crop damage from freezing, formation, and dispersion, among other things. Studying the various characteristics of the nanofluids in various geometries has garnered a lot of attention lately. Fluids with additional nanoscale particles are referred to as nanofluids. Water, ethylene glycol, and kerosene oil are examples of common heat transfer fluids that are poor heat conductors. That is, the development of energy-saving heat transfer equipment for power supply depends largely on these heating and cooling fluids. These fluids are mixed with conductive metal nanoparticles to improve and regulate thermal conductivity. Nanofluids are used in fuel cells, medicinal procedures, microelectronics, and other fields. Many researchers interested in the problem of nanofluid flow fields have done a lot of research in this field, as Sreenivasa Somireddy Reddy et al. [1] investigate

\*Corresponding author email : haedirnomoos@gmail.com



the effects of an inclined magnetic field on heat and mass transfer in turbulent squeeze flow of a viscoelastic fluid.

With an upper-convicted Maxwell model. Abdullah Dawaret et al. [2] the considered problem is solved by two analytical methods called IRPSM, and the homotopy perturbation method (HPM) for a squeezing flow between two parallel plates. Domairry and Hatami [3] looked at the flow analysis between two parallel plates of *Cu*-water squeezing nanofluid. The problem of very efficient, intelligent techniques have been used to solve the fourth-order nonlinear ordinary differential equations arising from squeezing unsteady nanofluid flow by Ahcene Nouar et al. [4]. Khan et al. [5] studied the squeezing flow of *Cu*-water or *Cu*-kerosene nanofluid between two parallel plates while accounting for velocity slip and viscous dissipation. Study approximate analytical unsteady flow and heat transfer analysis of CNTs nanofluid over stretching sheet for the improvement of heat assignment ratio by Ali Rehman et al. [6]. Additionally, Hassan et al. [7] the problem of physical flow on a compressed fluid is solved in Parallel plates through a porous Darcy channel when the fluid moves as a result of the compression of the upper plate towards the expanding lower plate. The problem of compressing fluid flow has been the subject of certain basic research [8–11], where findings show that the heat transfer rate reduces with an excess in the squeezing parameter, it increases with an excess in the nanoparticle volume fraction. The presence of the magnetic field in the imposed problems is important and has received much attention from researchers, such as Eid [12] who looked at the MHD mixed convection of nanofluid flow across an exponentially extending sheet when chemical reactions and heat generation-absorption effects filled up a porous medium. The effects of a magnetic field and heat generation absorption on the passage of a non-Newtonian nanofluid across a permeable stretched surface with suction- injection were investigated by Eid and Mahny [13]. Khaled Al-Farhan et al. [14] Numerical study of natural convective heat transfer of partially heated tall rectangular cavity filled with ( $Al_2O_3$ -water) nanofluid. Focuses Yahaya Shagaiya Daniel et al. [15] on the effects of suction as well as thermal radiation, chemical reaction, viscous dissipation, and Joule heating on a two-dimensional natural convective flow of unsteady electrical magnetohydrodynamics (MHD) nanofluid over a linearly permeable stretching sheet. This improvement involves combining the proposed method (q-HAM) with the Laplace transform supported by convolution theory and the Padé approximation [16- 18]. The effect was studied physical parameters on the velocity and temperature distributions for heat transfer of unsteady two-dimensional squeezing flow of a Casson fluid between parallel circular plates is studied [19]. The table below describes all the symbols used in this study.

### **Nomenclature**

|                    |                                       |               |                                     |
|--------------------|---------------------------------------|---------------|-------------------------------------|
| $\tilde{p}$        | fluid pressure                        | $B_0$         | electromagnetic induction           |
| $C_{p_{nf}}$       | specific heat of nanofluids           | $\varphi$     | angle of inclination magnetic field |
| $\hat{\rho}_{nf}$  | density of nanofluids                 | $Ha$          | Hartman number                      |
| $\check{\mu}_{nf}$ | dynamic viscosity of nanofluids       | $E_c$         | Eckert number                       |
| $\nu_{nf}$         | kinematic viscosity of nanofluids     | $Pr$          | Prandtl number                      |
| $\sigma_{nf}$      | electrical conductivity of nanofluids | $w$           | Nanoparticle volume fraction        |
| $k_{nf}(T)$        | variable thermal conductivity         | $S$           | squeeze number                      |
| $\sigma_f$         | base fluid                            | $\varepsilon$ | thermal conductivity                |

The principal interest and aim here is to investigate the effects of an inclined magnetic field, variable thermal conductivity, squeeze number, angle of inclination, and viscous dissipation on the velocity and temperature distributions for time-dependent heat transfer nanofluid flow between two parallel plates. Additionally, the effects of different kinds of nanoparticles such as *Au*, and  $Fe_3O_4$  are also studied. The governing equations resulting from transformations are resolved using the q-homotopy analysis approach and BVP4C. This means the proposed problem was first analytically solved and then numerically solved. Generally, the main of the present work is to improve the solutions of the q-HAM by combining the proposed method with the Laplace transform supported by convolution theory and the Padé approximation. Features of various flow parameters on Nusselt number and skin friction are discussed. Finally, solution results for LTCP-q-HAM are represented

in several tables and graphs and compared with BVP4C. The thermophysical properties of different nanoparticles are shown in Table (1) as follows:

**Table 1.** Thermophysical properties of pure water and nanoparticles.

| Material                           | $\rho(kg/m^3)$ | $C_p(j/kgk)$ | $k(W/mk)$ | $\sigma(Sm^{-1})$  |
|------------------------------------|----------------|--------------|-----------|--------------------|
| Clearwater                         | 997.1          | 4179         | 0.613     | 0.05               |
| <i>Au</i>                          | 19300          | 129          | 318       | $4.52 \times 10^7$ |
| <i>Fe<sub>3</sub>O<sub>4</sub></i> | 5180           | 670          | 9.7       | $0.74 \times 10^6$ |

## 2. Mathematical Formulation:

An unsteady two-dimensional heat transfer of a squeezing flow nanofluid between two infinitely extended parallel plates is under consideration with the effect of slip velocity. This formulation modeling applied a variable strength transverse magnetic field perpendicular to both plates. The definition of distance between two plates  $h(t) = \pm l(1 - \varphi t)^{1/2}$ , where  $l$  is the starting point ( $t = 0$ ).

This system has several physical properties, including incompressible flow, no chemical reaction, and viscous dissipation effects are maintained. Figure (1) explains the geometric model of the imposed problems. The following are the governing equations that depict the flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$\left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{1}{\hat{\rho}_{nf}} \frac{\partial \tilde{p}}{\partial x} + \frac{\check{\mu}_{nf}}{\hat{\rho}_{nf}} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\sigma_{nf}}{\hat{\rho}_{nf}} B_0^2 \sin^2 \varphi u, \quad (2)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{(\hat{\rho}c_p)_{nf}} \frac{\partial}{\partial y} \left( k_{nf}(T) \frac{\partial T}{\partial y} \right) + \frac{\check{\mu}_{nf}}{(\hat{\rho}c_p)_{nf}} \left( 4 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 \right), \quad (3)$$

The boundary conditions are:

$$\text{When } y \rightarrow 0: \frac{\partial u}{\partial y} = 0, v = 0, \text{ and } \frac{\partial T}{\partial y} = 0, \quad (4)$$

$$\text{When } y \rightarrow h(t): v_w = \frac{dh}{dt}, \text{ and } T = T_H, \quad (5)$$

where  $\tilde{p}$  is the fluid pressure,  $C_{p_{nf}}$  is the effective specific heat of the nanofluid at constant pressure,  $u$  and  $v$  are the  $x$  and  $y$  directions velocities,  $\hat{\rho}_{nf}$  is the density of the nanofluid,  $\check{\mu}_{nf}$  is the effective dynamic viscosity of the nanofluid,  $\nu_{nf}$  is the effective kinematic viscosity of the nanofluid,  $\sigma_{nf}$  is the electrical conductivity of the nanofluid,  $k_{nf}(T)$  is the variable thermal conductivity of the nanofluid,  $B_0$  is the electromagnetic induction, and  $\varphi$  is the angle of inclination of the applied magnetic field. The formulas for  $\hat{\rho}_{nf}$ ,  $\check{\mu}_{nf}$ ,  $\nu_{nf}$ ,  $C_{p_{nf}}$ ,  $\sigma_{nf}$  and  $k_{nf}(T)$  can be seen as follows [14, 15], respectively:

$$\begin{aligned} \hat{\rho}_{nf} &= \hat{\rho}_f(1 - w) + \hat{\rho}_s w, \quad \check{\mu}_{nf} = \frac{\check{\mu}_f}{(1-w)^{2.5}}, \quad (\hat{\rho}c_p)_{nf} = (1 - w)(\hat{\rho} \cdot c_p)_f + w(\hat{\rho} \cdot c_p)_s, \\ \frac{k_{nf}}{k_f} &= \frac{(k_s + 2k_f) - 2w(k_f - k_s)}{(k_s + 2k_f) + 2w(k_f - k_s)}, \quad \frac{\sigma_{nf}}{\sigma_f} = 1 + \frac{3\left(\frac{\sigma_s}{\sigma_f} - 1\right)w}{\left(\frac{\sigma_s}{\sigma_f} + 2\right) - \left(\frac{\sigma_s}{\sigma_f} - 1\right)w}, \quad k_{nf}(T) = k_{nf} \left[ 1 + \varepsilon \frac{T}{T_H} \right] \end{aligned} \quad (6)$$

Where,  $\sigma_f$  is the base fluid,  $\sigma_s$  denote the solid nanoparticles,  $\varepsilon$  is the constant temperature profile, and  $w$  is the volume fraction of nanoparticles. The values of dimensionless parameters are introduced as follows:

$$\tau = \frac{y}{l(1-\varphi t)^{1/2}}, \quad u = \frac{\varphi x}{2(1-\varphi t)} \frac{d\bar{f}}{d\tau}, \quad v = \frac{\varphi x}{2(1-\varphi t)^{1/2}} \bar{f}, \quad T = \bar{g} T_H. \quad (7)$$

By incorporating the terms of Equation (7) into Equations (2) and (3), the resulting formulation is obtained as follows

$$\frac{d^4 \bar{f}}{d\tau^4} - S.A_1.(1-w)^{2.5} \left[ \tau \frac{d^3 \bar{f}}{d\tau^3} + 3 \frac{d^2 \bar{f}}{d\tau^2} + \frac{d\bar{f}}{d\tau} \frac{d^2 \bar{f}}{d\tau^2} - \bar{f} \frac{d^3 \bar{f}}{d\tau^3} \right] - (1-w)^{2.5} Ha A_4 \sin^2(\varphi) \frac{d^2 \bar{f}}{d\tau^2} = 0 \quad (8)$$

$$(1 + \varepsilon \bar{g}) \frac{d^2 \bar{g}}{d\tau^2} + \varepsilon \left( \frac{d\bar{g}}{d\tau} \right)^2 + \frac{S.Pr A_2}{A_3} \left( \bar{f} \frac{d\bar{g}}{d\tau} - \tau \frac{d\bar{g}}{d\tau} \right) + \frac{Pr Ec}{A_3(1-w)^{2.5}} \left[ \left( \frac{d^2 \bar{f}}{d\tau^2} \right)^2 + 4\delta^2 \left( \frac{d\bar{f}}{d\tau} \right)^2 \right] = 0, \quad (9)$$

with:

$$A_1 = (1-w) + w \cdot \frac{\hat{\rho}_s}{\hat{\rho}_f}, A_2 = (1-w) + w \cdot \frac{(\hat{\rho} C_p)_s}{(\hat{\rho} C_p)_f}, A_3 = \frac{k_{nf}}{k_f} = \frac{(k_s + 2k_f) - 2w(k_f - k_s)}{(k_s + 2k_f) + 2w(k_f - k_s)},$$

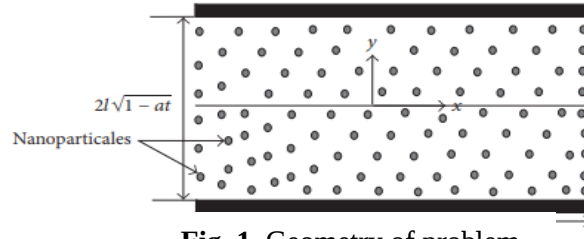
$$A_4 = \left[ 1 + \frac{3 \left( \frac{\sigma_s - 1}{\sigma_f} \right) w}{\left( \frac{\sigma_s + 2}{\sigma_f} \right) - \left( \frac{\sigma_s - 1}{\sigma_f} \right) w} \right]. \quad (10)$$

The Squeeze, Prandtl, Eckert, and Hartmann numbers are introduced as:

$$S = \frac{\varphi l^2}{2v_f}, \quad P_r = \frac{\check{\mu}_f \cdot (\hat{\rho} C_p)_f}{\hat{\rho}_f k_f}, \quad E_c = \frac{\hat{\rho}_f}{(\hat{\rho} C_p)_f} \left( \frac{\varphi x}{2(1-\varphi t)} \right)^2, \quad Ha = \frac{2\sigma_f B_0^2 (h(t))^2}{\hat{\rho}_f \check{\mu}_f}, \quad \delta = \frac{1}{x}.$$

The modified boundary conditions are

$$\bar{f}(0) = 0, \quad \frac{d^2 \bar{f}(0)}{d\tau^2} = 0, \quad \frac{d\bar{g}(0)}{d\tau} = 0, \quad \frac{d\bar{f}(1)}{d\tau} = 0, \quad \bar{f}(1) = 1, \quad \frac{d\bar{g}(1)}{d\tau} = 1. \quad (11)$$



**Fig .1.** Geometry of problem

The physical quantities of the skin friction coefficient and Nusselt number are explained as:

$$C_f = \frac{\check{\mu}_{nf} \left( \frac{\partial u}{\partial y} \right)_{y=h(t)}}{\hat{\rho}_{nf} v_w^2}, \quad N_u = \frac{-lk_{nf} \left( \frac{\partial T}{\partial y} \right)_{y=h(t)}}{k T_H},$$

The expression derived from the terms of Equation (8) is given as follows:

$$C_f^* = \frac{l^2}{x^2(1-\varphi t)} Re_x C_f = A_1(1-w)^{2.5} \bar{f}''(1), \quad N_u^* = \sqrt{1-\varphi t} N_u = -A_3 \bar{g}(1).$$

### 3. Basic Idea of q-Homotopy Analysis Method

The technique known as q-homotopy analysis is based on topological homotopy and conventional perturbation. This method yields an approximate solution that approaches the exact solution. The non-linear differential equation can be expressed in the following way to show the fundamental ideas of q-HAM:

$$D(\bar{f}) + U(\bar{f}) + N(\bar{f}) - g(\tau) = 0, \quad \tau \in \Omega \quad \mathcal{K}(\bar{f}(\tau), \frac{d\bar{f}}{d\tau}) = 0; \tau \in \Gamma \quad (12)$$

Where the boundaries of the field  $\Omega$  are displayed by  $\Gamma$ ,  $g(\tau)$  is a known analytic function,  $\bar{f}$  denotes the unknown function,  $\mathcal{K}$  denotes the boundary operator.  $D$  and  $U$  refer to the linear differential operator such that its order is less than  $D$ ,  $N$  is the general non-linear differential operator. The

definition of the homotopy in the homotopy perturbation technique is  $\bar{f}: \mathbb{R} \times [0, \frac{1}{n}] \rightarrow \mathbb{R}$  and constructed as

$$H(\bar{f}, q) = (1 - nq)[D(\bar{f}(\tau, q)) - D(\bar{f}_0)] - qhH(\tau)[D(\bar{f}(\tau)) + U(\bar{f}(\tau)) + N(\bar{f}(\tau)) - g(\tau)] = 0 \quad (13)$$

Here,  $H(\tau)$  indicates a non-zero auxiliary function,  $h \neq 0$  is an auxiliary parameter,  $q \in [0, \frac{1}{n}]$ ,  $n \geq 1$  indicates the so-called embedded parameter, and  $\bar{f}_0$  is an initial approximation of Equation (12) that precisely fulfills the boundary requirements.  $D$  is an acceptable auxiliary linear operator. The generic form of Equation (13) is as follows: when  $q = 0$ ,  $q = \frac{1}{n}$  is substituted, and  $H(\tau) = 1$ .

$$H(\bar{f}, 0) = D(\bar{f}(\tau, 0)) - D(\bar{f}_0), \quad (14)$$

$$H(\bar{f}, \frac{1}{n}) = \frac{h}{n}[D(\bar{f}(\tau, \frac{1}{n})) + U(\bar{f}(\tau, \frac{1}{n})) + N(\bar{f}(\tau, \frac{1}{n})) - g(\tau)], \quad (15)$$

The deformation of topology is known as  $\bar{f}(\tau, q)$  moves from  $\bar{f}_0(\tau)$  to  $\bar{f}(\tau)$  as moves from 0 to  $\frac{1}{n}$ .

As well,  $[D(\bar{f}(\tau, q)) - D(\bar{f}_0)]$  and  $[D(\bar{f}(\tau)) + U(\bar{f}(\tau)) + N(\bar{f}(\tau))]$  are referred to as homotopic. The power series that we obtain after solving Equation (13) is as follows:

$$\bar{f}(\tau, q) = \sum_{m=0}^{\infty} \bar{f}_m(\tau) q^m. \quad (16)$$

The correct solutions for the coefficients  $\bar{f}_m(\tau)$  in Equation (16) can be obtained using the homotopy deformation equations [16]. Consequently, it is simple to get an approximate analytical solution to Equation (12) as

$$\bar{f}(\tau) = \lim_{q \rightarrow \frac{1}{n}} \bar{f}(\tau, q) = \sum_{m=0}^{\infty} \bar{f}_m(\tau) \left(\frac{1}{n}\right)^m, \quad (17)$$

where

$$\bar{f}_m(\tau) = \frac{1}{m!} \left. \frac{d^m \bar{f}(\tau, q)}{dq^m} \right|_{q=0}. \quad (18)$$

Equation (13) is derived  $m$  times concerning  $q$  with by putting  $q = 0$  with the results by  $m!$ . The introduction of the vector is  $\vec{\bar{f}}_m(\tau) = \{\bar{f}_0(\tau), \bar{f}_1(\tau), \bar{f}(\tau), \dots, \bar{f}_m(\tau)\}$ . The  $m^{th}$ -order deformation equation can be defined by:

$$\vec{\bar{f}}_m(\tau) = \delta_m \vec{\bar{f}}_{m-1}(\tau) + hH(\tau)D^{-1}[R_m(\vec{\bar{f}}_{m-1}(\tau))], \quad (19)$$

where:

$$R_m(\vec{\bar{f}}_{m-1}(\tau)) = \frac{1}{(m-1)!} \left. \frac{d^{m-1}[D(\bar{f}(\tau, q)) + U(\bar{f}(\tau, q)) + N(\bar{f}(\tau, q)) - g(\tau)]}{dq^{m-1}} \right|_{q=0}, \quad (20)$$

And

$$\delta_m = \begin{cases} 0 & m \leq 1 \\ n & \text{otherwise} \end{cases}, \quad (21)$$

It is important to have linear boundary conditions for the proposed problem, the nonlinear Equation (19) is given  $\vec{\bar{f}}_m(\tau)$  for  $m \geq 1$ . In comparison to a typical HAM, the operator  $n$  allows for substantially faster convergence or even increases the probability of convergence. It should be noted that Equation (13) can be used to get the standard HAM for  $n = 1$ .

#### 4. Padé Approximation

Padé Approximate is the best approximation of the function by a rational function. The following is the rational function defined by the power series:

$$\bar{f}(\tau) = \sum_i^{\infty} c_i x^i. \quad (22)$$

A rational function  $\bar{f}(\tau)$  has a Padé approximation, which may be expressed in notation using the following form:

$$\bar{f}(\tau) = \frac{P_\ell(\tau)}{Q_k(\tau)}, \quad (23)$$

According to their definition,  $P_\ell$  is a polynomial of degree at most  $\ell$  and  $Q_\kappa$  is a polynomial of degree at most  $\kappa$  as

$$P_\ell(\tau) = \sum_{j=0}^{\ell} a_j \tau^j, \quad (24)$$

$$Q_\kappa(\tau) = \sum_{i=0}^{\kappa} b_i \tau^i, \quad (25)$$

Substituting Equations (22), (24), and (25) into Equation (23), yield

$$c_0 \tau^0 + c_1 \tau^1 + c_2 \tau^2 + \dots = \frac{a_0 \tau^0 + a_1 \tau^1 + a_2 \tau^2 + a_3 \tau^3 + \dots + a_\ell \tau^\ell}{b_0 \tau^0 + b_1 \tau^1 + b_2 \tau^2 + b_3 \tau^3 + \dots + b_\kappa \tau^\kappa} + O(\tau^{\ell+\kappa+1}), \quad (26)$$

Where  $b_0 = 1$ , Equation (26) is rearranged to take on the form

$$(c_0 \tau^0 + c_1 \tau^1 + c_2 \tau^2 + \dots)(b_0 \tau^0 + b_1 \tau^1 + b_2 \tau^2 + b_3 \tau^3 + \dots + b_\kappa \tau^\kappa) = a_0 \tau^0 + a_1 \tau^1 + a_2 \tau^2 + a_3 \tau^3 + \dots + a_\ell \tau^\ell + O(\tau^{\ell+\kappa+1}), \quad (27)$$

The following equations can be obtained from Equation (27).

$$c_0 = a_0,$$

$$c_0 b_1 + c_1 = a_1,$$

$$c_0 b_2 + c_1 b_1 + c_2 = a_2,$$

$$\vdots$$

$$c_0 b_\kappa + c_1 b_{\kappa-1} + c_2 b_{\kappa-2} + \dots + c_\kappa = a_\ell.$$

And

$$c_{\ell+1} + c_\ell b_1 + c_{\ell-1} b_2 + \dots + c_{\ell-\kappa+1} b_\kappa = 0,$$

$$c_{\ell+2} + c_{\ell+1} b_1 + c_\ell b_2 + \dots + c_{\ell-\kappa+2} b_\kappa = 0,$$

$$\vdots$$

$$c_{\ell+\kappa} + c_{\ell+\kappa-1} b_1 + c_{\ell+\kappa-2} b_2 + \dots + c_\ell b_\kappa = 0.$$

The Padé approximation of the rational function [17] can be obtained by solving the above system; we obtain the unknown values that are substituted into Equation (26).

## 5. The Basic Idea of LTCP-q-HAM

This section illustrates the hybrid methodology that combines the convolution theory-supported Laplace transform with the q-homotopy analysis method and Padé approximation. By rewriting Equation (12) assuming that  $D = \frac{d^n}{d\tau^n}$  as the operator's definition, which is as follows:

Considering the effects of the Laplace transforms on both sides of Equation (28), as follows:

$$\frac{d^n \bar{f}(\tau)}{d\tau^n} + U(\bar{f}(\tau)) + N(\bar{f}(\tau)) - g(\tau) = 0 \quad (28)$$

$$\mathcal{L} \left[ \frac{d^n \bar{f}(\tau)}{d\tau^n} + U(\bar{f}(\tau)) + N(\bar{f}(\tau)) - g(\tau) \right] = 0, \quad (29)$$

From the properties of the Laplace transform for Equation (29), the following outcomes are obtained:

$$s^n \mathcal{L}[\bar{f}(\tau)] - \sum_{\omega=0}^{n-1} s^{n-\omega-1} \bar{f}^{(\omega)}(0) + \mathcal{L} \left[ U(\bar{f}(\tau)) + N(\bar{f}(\tau)) - g(\tau) \right] = 0, \quad (30)$$

Rearranging Equation (30) yield in:

$$\mathcal{L}[\bar{f}(\tau)] - \frac{1}{s^n} \sum_{\omega=0}^{n-1} s^{n-\omega-1} \bar{f}^{(\omega)}(0) + \frac{1}{s^n} \mathcal{L} \left[ U(\bar{f}(\tau)) + N(\bar{f}(\tau)) - g(\tau) \right] = 0, \quad (31)$$

The Laplace transform properties can be applied, substituting  $\frac{1}{s^n} = \frac{\mathcal{L}[\tau^{n-1}]}{(n-1)!}$  in Equation (31); we get

$$\mathcal{L}[\bar{f}(\tau)] - \frac{1}{(n-1)!} \mathcal{L}[\tau^{n-1}] \sum_{\omega=0}^{n-1} s^{n-\omega-1} \bar{f}^{(\omega)}(0) + \frac{1}{(n-1)!} \mathcal{L}[\tau^{n-1}] \times \mathcal{L} \left[ U(\bar{f}(\tau)) + N(\bar{f}(\tau)) - g(\tau) \right] = 0, \quad (32)$$

Where Equation (32) can be written as follows:

$$\mathcal{L}[\bar{f}(\tau)] - \frac{1}{(n-1)!} \mathcal{L}[\tau^{n-1}] \sum_{\omega=0}^{n-1} s^{n-\omega-1} \bar{f}^{(\omega)}(0) + \mathcal{L} \left[ \frac{\tau^{n-1}}{(n-1)!} * \left( U(\bar{f}(\tau)) + N(\bar{f}(\tau)) - g(\tau) \right) \right] = 0, \quad (33)$$

The operation  $*$  is defined by:

- $\mathcal{L}[\bar{f}(\tau) * \bar{g}(\tau)] = \mathcal{L}[\bar{f}(\tau) \times \bar{g}(\tau)],$
- $\bar{f}(\tau) * \bar{g}(\tau) = \int_0^\tau \bar{f}(\tau - \xi) \bar{g}(\xi) d\xi,$

By applying the convolution theory of Equation (33), the following results can be become:

$$\mathcal{L}[\bar{f}(\tau)] - \frac{\mathcal{L}[\tau^{n-1}]}{(n-1)!} \sum_{\omega=0}^{n-1} s^{n-\omega-1} \bar{f}^{(\omega)}(0) + \mathcal{L} \left[ \int_0^\tau \frac{(\tau-\xi)^{n-1}}{(n-1)!} (U(\bar{f}(\tau)) + N(\bar{f}(\tau)) - g(\tau)) d\xi \right] = 0. \quad (34)$$

Hence, the nonlinear operator can be displayed as follows:

$$B[\bar{f}(\tau, q)] = \mathcal{L}[\bar{f}(\tau)] - \frac{\mathcal{L}[\tau^{n-1}]}{(n-1)!} \sum_{\omega=0}^{n-1} s^{n-\omega-1} \bar{f}^{(\omega)}(0) + \mathcal{L} \left[ \int_0^\tau \frac{(\tau-\xi)^{n-1}}{(n-1)!} \times (U(\bar{f}(\tau)) + N(\bar{f}(\tau)) - g(\tau)) d\xi \right]. \quad (35)$$

Now, taking the first term of Equation (19) Laplace transforms the results into are

$$\mathcal{L}[\bar{f}_m(\tau) - \delta_m \bar{f}_{m-1}(\tau)] = hqH(\tau)[R_m(\bar{f}_{m-1}(\tau))], \quad (36)$$

The above equation, after taking the inverse of the Laplace transform, has the following formula:

$$\bar{f}_m(\tau) = \delta_m \bar{f}_{m-1}(\tau) + hqH(\tau)\mathcal{L}^{-1}[R_m(\bar{f}_{m-1}(\tau))], \quad (37)$$

where,

$$R_m(\bar{f}_{m-1}(\tau)) = \frac{1}{(m-1)!} \frac{d^{m-1}(B[\bar{f}(\tau, q)])}{dq^{m-1}} \Big|_{q=0},$$

Pade' approximation  $P_{\kappa}^{\ell}$  of Equation (37)

$$P_{\kappa}^{\ell}[\bar{f}_m(\tau)] = \delta_m \bar{f}_{m-1}(\tau) + hqH(\tau)\mathcal{L}^{-1}[R_m(\bar{f}_{m-1}(\tau))], \quad (38)$$

## 6. The Applications of LTCP-q-HAM

The nonlinear ordinary differential Equations (9) and (10) are solved by LTCP-q-HAM and q-HAM to obtain an approximate analytical solution. The following is an explanation of these applications:

$$\frac{d^4 \bar{f}}{d\tau^4} - S.A_1.(1-w)^{2.5} \left[ \tau \frac{d^3 \bar{f}}{d\tau^3} + 3 \frac{d^2 \bar{f}}{d\tau^2} + \frac{d\bar{f}}{d\tau} \frac{d^2 \bar{f}}{d\tau^2} - \bar{f} \frac{d^3 \bar{f}}{d\tau^3} \right] - (1-w)^{2.5} HaA_4 \sin^2(\varphi) \frac{d^2 \bar{f}}{d\tau^2} = 0 \quad (39)$$

$$(1 + \varepsilon \bar{g}) \frac{d^2 \bar{g}}{d\tau^2} + \varepsilon \left( \frac{d\bar{g}}{d\tau} \right)^2 + \frac{S.PrA_2}{A_3} (\bar{f} \frac{d\bar{g}}{d\tau} - \tau \frac{d\bar{g}}{d\tau}) + \frac{PrEc}{A_3(1-w)^{2.5}} \left[ \left( \frac{d^2 \bar{f}}{d\tau^2} \right)^2 + 4\delta^2 \left( \frac{d\bar{f}}{d\tau} \right)^2 \right] = 0, \quad (40)$$

The assumptions of the initial conditions are:

$$\bar{f}_0(\tau) = \Gamma_0 + \Gamma_1 \tau + \Gamma_2 \frac{\tau^2}{2!} + \Gamma_3 \frac{\tau^3}{3!}, \quad (41)$$

$$\bar{g}_0(\tau) = \Psi_0 + \Psi_1 \tau, \quad (42)$$

Where  $\bar{f}(0) = \Gamma_0$ ,  $\bar{f}'(0) = \Gamma_1$ ,  $\bar{f}''(0) = \Gamma_2$ ,  $\bar{f}'''(0) = \Gamma_3$ ,  $\bar{g}(0) = \Psi_0$ ,  $\bar{g}'(0) = \Psi_1$ ,

By using Equation (11) and Equations (41) and (42) the groups of initial condition are readily found, as follows:

$$\bar{f}_0(\tau) = \Gamma_1\tau + \Gamma_3 \frac{\tau^3}{3!}, \quad (43)$$

$$\bar{g}_0(\tau) = \Psi_0, \quad (44)$$

From Adomain polynomials, a nonlinear operator is defined as follows:

$$R_m(\bar{f}_{m-1}(\tau)) = \frac{d^4 \bar{f}_{m-1}}{d\tau^4} - S.A_1.(1-w)^{2.5}[\tau \frac{d^3 \bar{f}_{m-1}}{d\tau^3} + 3 \frac{d^2 \bar{f}_{m-1}}{d\tau^2} + \sum_{z=0}^{m-1} \sum_{s=0}^z \frac{d}{d\tau} \bar{f}_s \frac{d^2}{d\tau^2} \bar{f}_{m-1-z}] - \sum_{z=0}^{m-1} \sum_{s=0}^z \bar{f}_s \frac{d^3}{d\tau^3} \bar{f}_{m-1-z}] - (1-w)^{2.5} Ha A_4 \sin^2(\varphi) \frac{d^2 \bar{f}_{m-1}}{d\tau^2}, \quad m = 1, 2, \dots \quad (45)$$

$$R_m(\bar{g}_{m-1}(\tau)) = (1 + \varepsilon \bar{g}) \frac{d^2 \bar{g}_{m-1}}{d\tau^2} + \varepsilon (\sum_{z=0}^{m-1} \frac{d}{d\tau} \bar{g}_{m-1-z})^2 + \frac{S.PrA_2}{A_3} (\sum_{z=0}^{m-1} \sum_{s=0}^z \bar{f}_s \frac{d}{d\tau} \bar{g}_{m-1-z} - \tau \frac{d \bar{g}_{m-1}}{d\tau}) + \frac{PrEc}{A_3(1-w)^{2.5}} \left[ \left( \sum_{z=0}^{m-1} \frac{d^2}{d\tau^2} \bar{f}_{m-1-z} \right)^2 + 4\delta^2 \left( \sum_{z=0}^{m-1} \frac{d}{d\tau} \bar{f}_{m-1-z} \right)^2 \right], \quad m = 1, 2, \dots \quad (46)$$

Now, Equations (8) and (9) can be solved to get:

$$\bar{f}_m(\tau) = \delta_m \bar{f}_{m-1}(\tau) + hH(\tau) \int_0^\tau \int_0^\tau \int_0^\tau \int_0^\tau \left[ \frac{d^4 \bar{f}_{m-1}}{d\tau^4} - S.A_1.(1-w)^{2.5}[\tau \frac{d^3 \bar{f}_{m-1}}{d\tau^3} + 3 \frac{d^2 \bar{f}_{m-1}}{d\tau^2} + \sum_{z=0}^{m-1} \sum_{s=0}^z \frac{d}{d\tau} \bar{f}_s \frac{d^2}{d\tau^2} \bar{f}_{m-1-z}] - \sum_{z=0}^{m-1} \sum_{s=0}^z \bar{f}_s \frac{d^3}{d\tau^3} \bar{f}_{m-1-z}] - (1-w)^{2.5} Ha A_4 \sin^2(\varphi) \frac{d^2 \bar{f}_{m-1}}{d\tau^2} \right] d\tau, \quad m = 1, 2, \dots \quad (47)$$

$$\bar{g}_m(\tau) = \delta_m \bar{g}_{m-1}(\tau) + hH(\tau) \int_0^\tau \int_0^\tau \left\{ (1 + \varepsilon \bar{g}) \frac{d^2 \bar{g}_{m-1}}{d\tau^2} + \varepsilon \left( \sum_{z=0}^{m-1} \frac{d}{d\tau} \bar{g}_{m-1-z} \right)^2 + \frac{S.PrA_2}{A_3} \left( \sum_{z=0}^{m-1} \sum_{s=0}^z \bar{f}_s \frac{d}{d\tau} \bar{g}_{m-1-z} - \tau \frac{d \bar{g}_{m-1}}{d\tau} \right) + \frac{PrEc}{A_3(1-w)^{2.5}} \left[ \left( \sum_{z=0}^{m-1} \frac{d^2}{d\tau^2} \bar{f}_{m-1-z} \right)^2 + 4\delta^2 \left( \sum_{z=0}^{m-1} \frac{d}{d\tau} \bar{f}_{m-1-z} \right)^2 \right] \right\} d\tau, \quad m = 1, 2, \dots \quad (48)$$

The resulting solution has several iterations of analytical solutions that can be introduced by the following:

$$\bar{f}_1(\tau) = (-0.04166666665hHaA_4 \sin^2(\varphi)(1-w)^{2.5}\Gamma_3 - 0.04166666665hSA_1(1-w)^{2.5}\Gamma_1\Gamma_3 -$$

$$0.1250000000hSA_1(1-w)^{2.5}\Gamma_3)\tau^4 - 0.008333333336hSA_1(1-w)^{2.5}\Gamma_3^2\tau^5.$$

$$\bar{f}_2(\tau) = (-0.04166666665nhHaA_4 \sin^2(\varphi)(1-w)^{2.5}\Gamma_3 - 0.04166666665nhSA_1(1-w)^{2.5}\Gamma_1\Gamma_3 -$$

$$0.1250000000nhSA_1(1-w)^{2.5}\Gamma_3 - 0.1250000000h^2SA_1(1-w)^{2.5}\Gamma_3 - 0.04166666665h^2HaA_4$$

$$\sin^2(\varphi)(1-w)^{2.5}\Gamma_3 - 0.041666666666h^2SA_1(1-w)^{2.5}\Gamma_1\Gamma_3)\tau^4(-0.008333333336nhSA_1(1-w)^{2.5}\Gamma_3^2 - 0.0083333333336h^2SA_1(1-w)^{2.5}\Gamma_3^2)\tau^5 + (0.02083333334h^2S^2A_1^2\Gamma_3 + 0.01388888888h^2Ha^2A_4^2\sin^4(\varphi)\Gamma_3w^2 - \dots$$

⋮

$$\bar{g}_1(\tau) = \frac{1}{A_3(1-w)^{2.5}} [(2hP_rE_c\delta^2\Gamma_1^2 + 0.5hP_rE_c\delta^2\Gamma_3^2)\tau^2 + 1.333333333hP_rE_c\delta^2\Gamma_1\Gamma_3\tau^3 + 0.333333333hP_rE_c\delta^2\Gamma_3^2\tau^4].$$

$$\bar{g}_2(\tau) = \frac{1}{A_3(1-w)^{2.5}} (2nhP_rE_c\delta^2\Gamma_1^2 + 0.5nhP_rE_c\Gamma_3^2 + 0.5h^2P_rE_c\varepsilon\Psi_0\Gamma_3^2 + 2h^2P_rE_c\varepsilon\Psi_0\Gamma_1^2 +$$

$$0.5h^2P_rE_c\Gamma_3^2 + 2h^2P_rE_c\delta^2\Gamma_1^2)\tau^2 + \frac{1}{A_3(1-w)^{2.5}} (1.333333333nhP_rE_c\delta^2\Gamma_1\Gamma_3 +$$

$$1.333333333h^2P_rE_c\delta^2\Gamma_1\Gamma_3 + 1.333333333h^2P_rE_c\delta^2\varepsilon\Psi_0\Gamma_1\Gamma_3)\tau^3 + \frac{0.333333333nhP_rE_c\delta^2\Gamma_3^2}{A_3(1-w)^{2.5}}$$

$$- \frac{0.0833333332h^2SP_r^2E_cA_2\Gamma_3^2}{A_3^2(1-w)^{2.5}} - \frac{0.08333333330h^2P_rE_cA_4H\sin^2(\varphi)\Gamma_3^2}{A_3} -$$

$$\frac{0.08333333330h^2SP_rE_cA_1\Gamma_3^2}{A_3} + \frac{0.333333332h^2SP_r^2E_c\delta^2A_2\Gamma_1^3}{A_3^2(1-w)^{2.5}} + \dots$$

The remaining iterative components,  $\bar{f}_m(\tau)$  and  $\bar{g}_m(\tau)$ ,  $m = 3, 4, 5, 6, \dots$  can be acquired by resemblance. The following is the formulation for the series solution by q-HAM:

$$\left. \begin{aligned} \bar{f}(\tau, n; h) &\cong \bar{f}_m(\tau, n; h) = \sum_{i=0}^m u_i(\tau, n; h) \left(\frac{1}{n}\right)^i, \quad n = 1, 2, \dots \\ \bar{g}(\tau, n; h) &\cong \bar{g}_m(\tau, n; h) = \sum_{i=0}^m u_i(\tau, n; h) \left(\frac{1}{n}\right)^i, \quad n = 1, 2, \dots \end{aligned} \right\} \quad (49)$$

By using LTCP-q-HAM to find the analytical approximate solution by taking the Laplace transform to both sides of Equations (8) and (9), become

$$\mathcal{L} \left[ \frac{d^4 \bar{f}}{d\tau^4} \right] - S.A_1.(1-w)^{2.5} \mathcal{L} \left[ \tau \frac{d^3 \bar{f}}{d\tau^3} + 3 \frac{d^2 \bar{f}}{d\tau^2} + \frac{d\bar{f}}{d\tau} \frac{d^2 \bar{f}}{d\tau^2} - \bar{f} \frac{d^3 \bar{f}}{d\tau^3} \right] - (1-w)^{2.5} Ha A_4 \sin^2(\varphi)$$

$$\mathcal{L} \left[ \frac{d^2 \bar{f}}{d\tau^2} \right] = 0, \quad (50)$$

$$\mathcal{L} \left[ \frac{d^2 \bar{g}}{d\tau^2} \right] + \mathcal{L} \left[ \varepsilon \bar{g} \frac{d^2 \bar{g}}{d\tau^2} + \varepsilon \left( \frac{d\bar{g}}{d\tau} \right)^2 + \frac{S.PrA_2}{A_3} \left( \bar{f} \frac{d\bar{g}}{d\tau} - \tau \frac{d\bar{g}}{d\tau} \right) + \frac{PrEc}{A_3(1-w)^{2.5}} \left[ \left( \frac{d^2 \bar{f}}{d\tau^2} \right)^2 + 4\delta^2 \left( \frac{d\bar{f}}{d\tau} \right)^2 \right] \right] = 0, \quad (51)$$

By implementing the differentiation property of the Laplace transform with the help of the initial condition from Equations (50) and (51), we get

$$\begin{aligned} \mathcal{L}[\bar{f}] - \left( \frac{\Gamma_1 s^2 + \Gamma_3}{s^4} \right) - \frac{1}{s^4} \mathcal{L}[S.A_1 \cdot (1-w)^{2.5} \left( \tau \frac{d^3 \bar{f}}{d\tau^3} + 3 \frac{d^2 \bar{f}}{d\tau^2} + \frac{d\bar{f}}{d\tau} \frac{d^2 \bar{f}}{d\tau^2} - \bar{f} \frac{d^3 \bar{f}}{d\tau^3} \right) \\ - (1-w)^{2.5} HaA_4 \end{aligned} \sin^2(\varphi) \frac{d^2 \bar{f}}{d\tau^2}] = 0, \quad (52)$$

$$\begin{aligned} \mathcal{L}[\bar{g}] - \frac{\Psi_0}{s} + \frac{1}{s^2} \mathcal{L}\{\varepsilon \bar{g} \frac{d^2 \bar{g}}{d\tau^2} + \varepsilon \left( \frac{d\bar{g}}{d\tau} \right)^2 + \frac{S.PrA_2}{A_3} (\bar{f} \frac{d\bar{g}}{d\tau} - \tau \frac{d\bar{g}}{d\tau}) + \frac{PrEc}{A_3(1-w)^{2.5}} \left[ \left( \frac{d^2 \bar{f}}{d\tau^2} \right)^2 + \right. \\ \left. 4\delta^2 \left( \frac{d\bar{f}}{d\tau} \right)^2 \right] \} = 0, \quad (53) \end{aligned}$$

From the Laplace transform's properties substituting  $\frac{1}{s^4} = \frac{1}{3!} \mathcal{L}[\tau^3]$  and  $\frac{1}{s^2} = \frac{1}{1!} \mathcal{L}[\tau]$  in Equations (52) and (53) respectively, we get the following results:

$$\begin{aligned} \mathcal{L}[\bar{f}] - \left( \frac{\Gamma_1 s^2 + \Gamma_3}{s^4} \right) - \frac{1}{3!} \mathcal{L}[\tau^3] \times \mathcal{L}[S.A_1 \cdot (1-w)^{2.5} \left( \tau \frac{d^3 \bar{f}}{d\tau^3} + 3 \frac{d^2 \bar{f}}{d\tau^2} + \frac{d\bar{f}}{d\tau} \frac{d^2 \bar{f}}{d\tau^2} - \bar{f} \frac{d^3 \bar{f}}{d\tau^3} \right) \\ - (1-w)^{2.5} \end{aligned} HaA_4 \sin^2(\varphi) \frac{d^2 \bar{f}}{d\tau^2}] = 0, \quad (54)$$

$$\begin{aligned} \mathcal{L}[\bar{g}] - \frac{\Psi_0}{s} + \frac{1}{1!} \mathcal{L}[\tau] \times \mathcal{L}\{\varepsilon \bar{g} \frac{d^2 \bar{g}}{d\tau^2} + \varepsilon \left( \frac{d\bar{g}}{d\tau} \right)^2 + \frac{S.PrA_2}{A_3} (\bar{f} \frac{d\bar{g}}{d\tau} - \tau \frac{d\bar{g}}{d\tau}) + \frac{PrEc}{A_3(1-w)^{2.5}} \left[ \left( \frac{d^2 \bar{f}}{d\tau^2} \right)^2 + \right. \\ \left. 4\delta^2 \left( \frac{d\bar{f}}{d\tau} \right)^2 \right] \} = 0, \quad (55) \end{aligned}$$

The convolution theory concept uses the last terms of Equation (54) and Equation (55), the outcome can be obtained following:

$$\begin{aligned} \mathcal{L}[\bar{f}] - \left( \frac{\Gamma_1 s^2 + \Gamma_3}{s^4} \right) - \mathcal{L}\left[ \frac{1}{3!} \tau^3 * (S.A_1 \cdot (1-w)^{2.5} \left( \tau \frac{d^3 \bar{f}}{d\tau^3} + 3 \frac{d^2 \bar{f}}{d\tau^2} + \frac{d\bar{f}}{d\tau} \frac{d^2 \bar{f}}{d\tau^2} - \bar{f} \frac{d^3 \bar{f}}{d\tau^3} \right) - \right. \\ \left. (1-w)^{2.5} HaA_4 \sin^2(\varphi) \frac{d^2 \bar{f}}{d\tau^2} \right] = 0, \quad (56) \end{aligned}$$

$$\begin{aligned} \mathcal{L}[\bar{g}] - \frac{\Psi_0}{s} + \mathcal{L}\left\{ \tau * \left( \varepsilon \bar{g} \frac{d^2 \bar{g}}{d\tau^2} + \varepsilon \left( \frac{d\bar{g}}{d\tau} \right)^2 + \frac{S.PrA_2}{A_3} (\bar{f} \frac{d\bar{g}}{d\tau} - \tau \frac{d\bar{g}}{d\tau}) + \frac{PrEc}{A_3(1-w)^{2.5}} \left[ \left( \frac{d^2 \bar{f}}{d\tau^2} \right)^2 + \right. \right. \right. \\ \left. \left. \left. + 4\delta^2 \left( \frac{d\bar{f}}{d\tau} \right)^2 \right] \right) \right\} = 0, \quad (57) \end{aligned}$$

From taking the inverse Laplace transform for both sides of Equations (56) and (57) and from q-HAM the following results are:

$$\bar{f}_m(\tau) = \delta_m \bar{f}_{m-1}(\tau) + h \mathcal{L}^{-1} \left( \mathcal{L}[\bar{f}_{m-1}] - \left( 1 - \frac{1}{n} \delta_m \right) \left( \frac{\Gamma_1 s^2 + \Gamma_3}{s^4} \right) \right) - \frac{h}{3!} \int_0^\tau ((\tau - \xi)^3 [S.A_1(1-w)^{2.5} \cdot$$

$$\left( \tau \frac{d^3 \bar{f}}{d\tau^3} + 3 \frac{d^2 \bar{f}}{d\tau^2} + \sum_{z=0}^{m-1} \sum_{s=0}^z \frac{d}{d\tau} \bar{f}_s \frac{d^2}{d\tau^2} \bar{f}_{m-1-z} \right) - \sum_{z=0}^{m-1} \sum_{s=0}^z \bar{f}_s \frac{d^3}{d\tau^3} \bar{f}_{m-1-z}] -$$

$$(1-w)^{2.5}HaA_4\sin^2(\varphi)\frac{d^2\bar{f}}{d\tau^2}\Big|_{\tau=\xi})d\xi, \quad (58)$$

$$\begin{aligned} \bar{g}_m(\tau) = & \delta_m \bar{g}_{m-1}(\tau) + h\mathcal{L}^{-1}\left(\mathcal{L}[\bar{g}_{m-1}] - \left(1 - \frac{1}{n}\delta_m\right)\left(\frac{\Psi_0}{s}\right)\right) + \int_0^\tau ((\tau - \\ & \xi)[\sum_{z=0}^{m-1} \sum_{s=0}^z \varepsilon \bar{g}_s \frac{d^2}{d\tau^2} \bar{g}_{m-1-z} \\ & + \varepsilon(\sum_{z=0}^{m-1} \frac{d}{d\tau} \bar{g}_{m-1-z})^2 + \frac{S.PrA_2}{A_3}(\sum_{z=0}^{m-1} \sum_{s=0}^z \bar{f}_s \frac{d}{d\tau} \bar{g}_{m-1-z} - \tau \frac{d\bar{g}_{m-1}}{d\tau}) + \frac{PrEc}{A_3(1-w)^{2.5}} \\ & [(\sum_{z=0}^{m-1} \frac{d^2}{d\tau^2} \bar{f}_{m-1-z})^2 + 4\delta^2(\sum_{z=0}^{m-1} \frac{d}{d\tau} \bar{f}_{m-1-z})^2] \Big|_{\tau=\xi})d\xi, \end{aligned} \quad (59)$$

$$\begin{aligned} \bar{f}_1(\tau) = & \Gamma_1\tau + \frac{1}{n}[(0.5h\Gamma_3 + 0.5\Gamma_3n)\tau^2 - 0.16666666667h\Gamma_3\tau^3 \\ & + (-0.04166666665hHaA_4\sin^2(\varphi) \end{aligned}$$

$$\begin{aligned} (1-w)^{2.5}\Gamma_3 - 0.04166666665hSA_1(1-w)^{2.5}\Gamma_1\Gamma_3 - 0.125hSA_1(1-w)^{2.5}\Gamma_3)\tau^4 - \\ 0.008333333350hSA_1(1-w)^{2.5}\Gamma_3^2\tau^5. \end{aligned}$$

$$\begin{aligned} \bar{f}_2(\tau) = & \Gamma_1\tau + \frac{1}{n^2}[(0.5000000001h^2\Gamma_3 + nh\Gamma_3 + 0.5000000000n^2\Gamma_3)\tau^2 \\ & + (-0.1666666667h^2\Gamma_3 \\ & - 0.3333333334nh\Gamma_3)\tau^3 + (-0.008333333331h^2HaA_4\sin^2(\varphi)(1-w)^{2.5}\Gamma_3 - \\ & 0.008333333330nhHaA_4\sin^2(\varphi)(1-w)^{2.5}\Gamma_3 - 0.008333333331h^2SA_1(1-w)^{2.5}\Gamma_1\Gamma_3 - \\ & 0.008333333330nhSA_1(1-w)^{2.5}\Gamma_1\Gamma_3 - 0.2500000000h^2SA_1(1-w)^{2.5}\Gamma_3 - \\ & 0.2500000000nhSA_1(1-w)^{2.5}\Gamma_3)\tau^4 + (0.0008333333350h^2HaA_4\sin^2(\varphi)(1-w)^{2.5}\Gamma_3 - \\ & 0.002500000002h^2SA_1(1-w)^{2.5}\Gamma_3^2 - \dots \\ & \vdots \end{aligned}$$

Now, take the Pade's approximation at the fourth iteration we get:

$$\begin{aligned} P_0^7(\bar{f}_4(\tau)) = & 1.445131944\tau - 0.001273801303\tau^2 - 0.3881113444\tau^3 - 0.004971623653\tau^4 \\ & - 0.04782596097\tau^5 - 0.001410484899\tau^6 - 0.001538728431\tau^7. \end{aligned}$$

$$\begin{aligned} \bar{g}_1(\tau) = & \Psi_0 + \frac{1}{nA_3(1-w)^{2.5}}[(2hP_rE_c\delta^2\Gamma_1^2 + 0.5hP_rE_c\delta^2\Gamma_3^2)\tau^2 + 1.333333334hP_rE_c\delta^2\Gamma_1\Gamma_3\tau^3 \\ & + \\ & 0.3333333330hP_rE_c\delta^2\Gamma_3^2\tau^4]. \end{aligned}$$

$$\bar{g}_2(\tau) = \Psi_0 + \frac{1}{A_3^2(1-w)^{2.5}n^2}[(2h^2P_rE_c\delta^2\varepsilon\Psi_0A_3\Gamma_1^2 + 2h^2P_rE_c\delta^2A_3\Gamma_1^2 + 4nhP_rE_c\delta^2A_3\Gamma_1^2 +$$

$$\begin{aligned}
& 0.5h^2P_rE_c\varepsilon\Psi_0A_3\Gamma_3^2 + 1.5h^2P_rE_cA_3\Gamma_3^2 + nhP_rE_cA_3\Gamma_3^2)\tau^2 + (1.333333334h^2P_rE_c\delta^2\varepsilon\Psi_0A_3\Gamma_1\Gamma_3 \\
& + \\
& 2.666666668h^2P_rE_c\delta^2A_3\Gamma_1\Gamma_3 + 2.666666668nhP_rE_c\delta^2A_3\Gamma_1\Gamma_3 - 0.333333334h^2P_rE_c\delta^2A_3\Gamma_3^2)\tau^3 + \\
& (-0.08333333330h^2P_rE_cA_3A_4Hasin^2(\varphi)(1-w)^{2.5}\Gamma_3^2 \\
& - 0.08333333330h^2P_rE_cSA_1A_3Hasin^2(\varphi) \\
& (1-w)^{2.5}\Gamma_1\Gamma_3^2 - 0.2499999999h^2P_rE_cSA_1A_3(1-w)^{2.5}\Gamma_3^2 + 0.3333333330h^2P_rE_c\delta^2SA_2\Gamma_1^3 \\
& - \dots
\end{aligned}$$

Now, take Pade's approximation at the fourth iteration we get:

$$\begin{aligned}
P_0^7(\bar{f}_4(\tau)) = & 1.276801013 + 0.1328106758\tau^2 - 0.1198231839\tau^3 - 0.03499795754\tau^4 - \\
& 0.1810308875\tau^5 - 0.02213432925\tau^6 - 0.05162533074\tau^7.
\end{aligned}$$

## 6. Results and Discussions

The impact of an inclined magnetic field and variable thermal conductivity on heat transfer of squeezing unsteady ferrofluid flow with nanoparticle materials  $Au$ , and  $Fe_3O_4$  has been examined, as well as different values for  $S, Ha, \delta, w, \varepsilon, P_r, E_c$ , and  $\varphi$  for velocity and temperature distributions. Tables (2) and (3) the convergence of values  $\Gamma_1$  and  $\Gamma_3$  for nanoparticle materials  $Au$ , and  $Fe_3O_4$  is illustrated. These tables show that the values of  $\Gamma_1$  are constant for q-HAM at the fourth approximation and for LTCP-q-HAM at the third approximation, while the values of  $\Gamma_3$  are constant for q-HAM at the fifth approximation but LTCP-q-HAM at the third approximation, meaning that the constants for LTCP-q-HAM are better and converge to the values of BVP4C. From Tables (4) and (5), it can be observed that the convergence of  $\Psi_0$ , where the new technique appears converges to the numerical approximation of q-HAM. The effect of the Nusselt number for nanoparticle materials is displayed in Tables (6)-(8). From these tables, it can be seen that the values of the Nusselt number for LTCP-q-HAM are more accurate than for q-HAM when  $\varphi = 5^\circ$  and  $\varphi = 12^\circ$  and it can be observed that the values of the BVP4C for  $\varphi = 5^\circ$  for all materials are equal; which means the effect of the materials is similar, while the values of the BVP4C for  $\varphi = 12^\circ$  are different meaning, these materials have the same impact for  $0 \leq \varphi \leq 12^\circ$  while inverse occurs when  $\varphi \geq 12^\circ$  Tables (9)-(11). These tables demonstrated the increasing of physical parameters led to an increment of the skin friction coefficient as well as the optimization of results through LTCP-q-HAM.

**Table 2.** The convergence of  $\Gamma_1, \Gamma_3$  for material  $Au$  when  $n = 2, h = -2, S = 0.5, Ha = 10, \delta = 0.1, w = 0, \varepsilon = 0, P_r = 0.1, E_c = 0.01, \varphi = 5^\circ$ .

| Approximates | q-HAM           |                  | LTCP-q-HAM      |                  | BVP4C           |                  |
|--------------|-----------------|------------------|-----------------|------------------|-----------------|------------------|
|              | $\Gamma_1$      | $\Gamma_3$       | $\Gamma_1$      | $\Gamma_3$       | $\Gamma_1$      | $\Gamma_3$       |
| 1 order      | 1.759799        | -1.272649        | 1.431209        | -1.690339        | 1.493209        | -3.092849        |
| 2 orders     | 1.742052        | -1.232354        | 1.432153        | -2.204750        | 1.485231        | -3.092581        |
| 3 orders     | 1.741697        | -1.231622        | <u>1.457024</u> | <u>-2.484290</u> | 1.485231        | -3.092581        |
| 4 orders     | <u>1.741707</u> | -1.231640        | <u>1.457024</u> | <u>-2.484290</u> | 1.485231        | -3.092581        |
| 5 orders     | <u>1.741707</u> | <u>-1.231641</u> | <u>1.457024</u> | <u>-2.484290</u> | <u>1.485245</u> | <u>-3.092593</u> |
| 6 orders     | <u>1.741707</u> | <u>-1.231641</u> | <u>1.457024</u> | <u>-2.484290</u> | <u>1.485245</u> | <u>-3.092593</u> |

**Table 3.** The convergence of  $\Gamma_1, \Gamma_3$  for material  $Fe_3O_4$  when  $n = 2, h = -2, S = 0.5, Ha = 10, \delta = 0.1, w = 0, \varepsilon = 0, P_r = 0.1, E_c = 0.01, \varphi = 5^\circ$ .

| Approximates | q-HAM      |            | LTCP-q-HAM      |                  | BVP4C      |            |
|--------------|------------|------------|-----------------|------------------|------------|------------|
|              | $\Gamma_1$ | $\Gamma_3$ | $\Gamma_1$      | $\Gamma_3$       | $\Gamma_1$ | $\Gamma_3$ |
| 1 order      | 1.759799   | -1.272649  | 1.431209        | -1.690339        | 1.493209   | -3.092849  |
| 2 orders     | 1.742052   | -1.232354  | 1.432153        | -2.204750        | 1.485231   | -3.092581  |
| 3 orders     | 1.741697   | -1.231622  | <u>1.457024</u> | <u>-2.484290</u> | 1.485231   | -3.092581  |

|          |                 |                  |                 |                  |                 |                  |
|----------|-----------------|------------------|-----------------|------------------|-----------------|------------------|
| 4 orders | <u>1.741707</u> | -1.231640        | <u>1.457024</u> | <u>-2.484290</u> | 1.485231        | -3.092581        |
| 5 orders | <u>1.741707</u> | <u>-1.231641</u> | <u>1.457024</u> | <u>-2.484290</u> | <u>1.485245</u> | <u>-3.092593</u> |
| 6 orders | <u>1.741707</u> | <u>-1.231641</u> | <u>1.457024</u> | <u>-2.484290</u> | <u>1.485245</u> | <u>-3.092593</u> |

**Table 4.** The convergence of  $\Psi_0$  for material *Au* when  $n = 2, h = -2, S = 0.5, Ha = 10, \delta = 0.1, w = 0, \varepsilon = 0, P_r = 0.1, E_c = 0.01, \varphi = 5^\circ$

| Approximates    | q-HAM           | LTCP-q-HAM      | BVP4C           |
|-----------------|-----------------|-----------------|-----------------|
| 1 order         | 1.000847        | 1.001446        | 1.000786        |
| 2 orders        | 1.001086        | 1.000058        | <u>1.000744</u> |
| 3 orders        | 1.001167        | 1.000732        | <u>1.000744</u> |
| 4 orders        | 1.001177        | <u>1.000686</u> | <u>1.000744</u> |
| <u>5 orders</u> | <u>1.001178</u> | <u>1.000686</u> | <u>1.000744</u> |
| <u>6 orders</u> | <u>1.001178</u> | <u>1.000686</u> | <u>1.000744</u> |

**Table 5.** The convergence of  $\Psi_0$  for material *Fe<sub>3</sub>O<sub>4</sub>* when  $n = 2, h = -2, S = 0.5, Ha = 10, \delta = 0.1, w = 0, \varepsilon = 0, P_r = 0.1, E_c = 0.01, \varphi = 5^\circ$

| Approximates | q-HAM           | LTCP-q-HAM      | BVP4C           |
|--------------|-----------------|-----------------|-----------------|
| 1 order      | 1.000847        | 1.001446        | 1.000786        |
| 2 order      | 1.001086        | 1.000058        | <u>1.000744</u> |
| 3 order      | 1.001167        | 1.000732        | <u>1.000744</u> |
| 4 order      | 1.001177        | <u>1.000686</u> | <u>1.000744</u> |
| 5 order      | <u>1.001178</u> | <u>1.000686</u> | <u>1.000744</u> |
| 6 order      | <u>1.001178</u> | <u>1.000686</u> | <u>1.000744</u> |

**Table 6.** Comparison of  $-\bar{g}'(1)$  between q-HAM and LTCP-q-HAM for  $n = 1.1, h = -0.9, S = 0.5, Ha = 50, \delta = 0.1, w = 0, \varepsilon = 0, \varphi = 5^\circ$ .

|       |       | <i>Au</i>     |               | <i>Fe<sub>3</sub>O<sub>4</sub></i> |               |               |
|-------|-------|---------------|---------------|------------------------------------|---------------|---------------|
| $P_r$ | $E_c$ | q-HAM         | LTCP-qHAM     | q-HAM                              | LTCP-qHAM     | BVP4C         |
| 0.5   | 1     | 1.52336273108 | 1.51350765323 | 1.52336273108                      | 1.51350766836 | 1.51717817687 |
| 1     | 1     | 3.01023336320 | 3.01384328341 | 3.01023336320                      | 3.01384331529 | 3.01461698607 |
| 2     | 1     | 5.87923137399 | 5.9384685834  | 5.87923137399                      | 5.9384685710  | 5.95200370035 |
| 5     | 1     | 13.7276557447 | 14.3119436884 | 13.7276557447                      | 14.3119436785 | 14.3334284523 |
| 1     | 0.5   | 1.50511668138 | 1.50692164676 | 1.50511668138                      | 1.50692165158 | 1.50730849306 |
| 1     | 1.2   | 3.61228003537 | 3.61661197376 | 3.61228003537                      | 3.61661197956 | 3.61754038329 |
| 1     | 2     | 6.02046672467 | 6.02768656316 | 6.02046672468                      | 6.02768656124 | 6.02923397215 |
| 1     | 5     | 15.0511668138 | 15.0692164832 | 15.0511668138                      | 15.0692165015 | 15.0730849303 |

**Table 7.** Comparison of  $-\bar{g}'(1)$  between q-HAM and LTCP-q-HAM with *Au* for  $n = 1.1, h = -1.26, S = 1, Ha = 100, \delta = 0.01, w = 0.01, \varepsilon = 0.001, \varphi = 12^\circ$

| $P_r$ | $E_c$ | q-HAM         | LTCP-q-HAM    | BVP4C         |
|-------|-------|---------------|---------------|---------------|
| 0.5   | 1     | 1.52259102292 | 1.52204656620 | 1.52220372687 |
| 1     | 1     | 3.00816104964 | 3.01366523122 | 3.02068120677 |
| 2     | 1     | 5.87295921439 | 5.92585684374 | 5.94944357501 |
| 5     | 1     | 13.6842385762 | 14.2039739812 | 14.2388987838 |
| 1     | 0.5   | 1.50436754053 | 1.50682930355 | 1.51033642106 |
| 1     | 1.2   | 3.60951787027 | 3.61640146283 | 3.62482146152 |
| 1     | 2     | 6.01402851945 | 6.02735758201 | 6.04139584288 |
| 1     | 5     | 15.0179198304 | 15.0686163924 | 15.1037397441 |

**Table 8.** Comparison of  $-\bar{g}'(1)$  between q-HAM and LTCP-q-HAM with *Fe<sub>3</sub>O<sub>4</sub>* for  $n = 1.1, h = -1.26, S = 1, Ha = 100, \delta = 0.01, w = 0.01, \varepsilon = 0.001, \varphi = 12^\circ$

| $P_r$ | $E_c$ | q-HAM         | LTCP-q-HAM    | BVP4C         |
|-------|-------|---------------|---------------|---------------|
| 0.5   | 1     | 1.53631722722 | 1.52472320542 | 1.52941430817 |
| 1     | 1     | 3.03386424379 | 3.03399863250 | 3.03449706503 |

|   |     |               |               |               |
|---|-----|---------------|---------------|---------------|
| 2 | 1   | 5.91780223471 | 5.94230315080 | 5.97477253772 |
| 5 | 1   | 13.7520776535 | 14.1458616836 | 14.2870599288 |
| 1 | 0.5 | 1.51723447977 | 1.51723788959 | 1.51724419543 |
| 1 | 1.2 | 3.64034698616 | 3.64087146653 | 3.64140063992 |
| 1 | 2   | 6.06531246732 | 6.0686064159  | 6.06902879604 |
| 1 | 5   | 15.1452175580 | 15.1482760616 | 15.1728313736 |

**Table 9.** The comparison of  $-\bar{f}''(1)$  for  $n = 3, h = -3, Ha = 10, P_r = 0.1, E_c = 0.01, \delta = 0.1, w = 0.02, \varepsilon = 0.01, \varphi = 5^\circ$

| $S$ | q-HAM          | LTCP-q-HAM     | BVP4C          |
|-----|----------------|----------------|----------------|
| 0   | 3.191698897084 | 4.721924867157 | 4.722056565409 |
| 0.5 | 4.870357728361 | 5.458861461486 | 5.171026835811 |
| 1.5 | 6.600915166112 | 6.221575690289 | 5.890627319671 |
| 2   | 7.218543891604 | 6.444418935022 | 6.180019541008 |
| 4   | 9.062755459888 | 6.938477789613 | 7.046600558228 |
| 6   | 10.33660050647 | 7.172804431174 | 7.646118574667 |

**Table 10.** The comparison of  $-\bar{f}''(1)$  for  $n = 3, h = -3, S = 1, P_r = 0.1, E_c = 0.01, \delta = 0.1, w = 0.02, \varepsilon = 0.01, \varphi = 5^\circ$

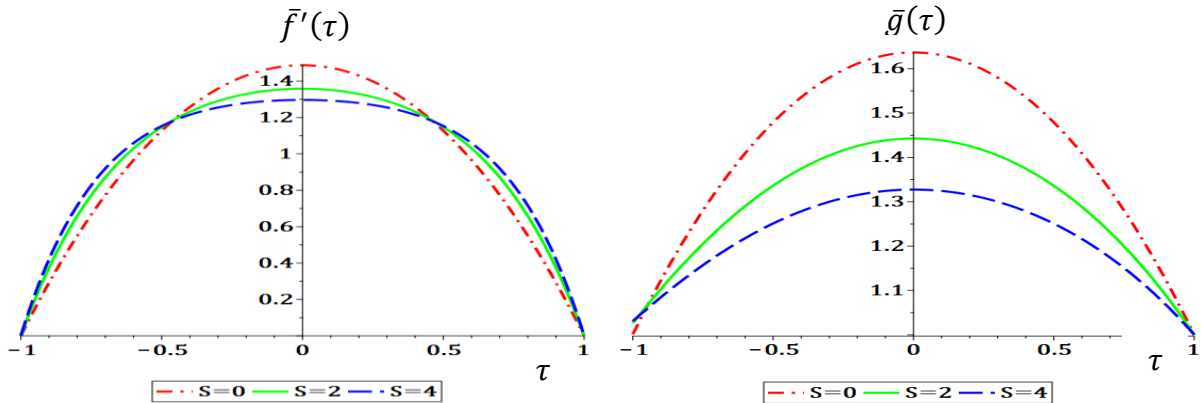
| $Ha$ | q-HAM          | LTCP-q-HAM     | BVP4C          |
|------|----------------|----------------|----------------|
| 0    | 5.534567108213 | 5.769317166121 | 5.413985749409 |
| 100  | 5.774608879712 | 5.868140840938 | 5.596733959683 |
| 200  | 6.000795490982 | 5.957919509285 | 5.774191735604 |
| 300  | 6.214964554771 | 6.039840619352 | 5.946748896744 |
| 350  | 6.318015097697 | 6.078170049160 | 6.031299626409 |
| 550  | 6.706817272438 | 6.216368597740 | 6.358887984981 |
| 750  | 7.063688377435 | 6.334356433487 | 6.671101452309 |

**Table 11.** The comparison of  $-\bar{f}''(1)$  for  $n = 3, h = -3, Ha = 50, S = 1, P_r = 0.1, E_c = 0.01, \delta = 0.1, \varepsilon = 0.01, \varphi = 5^\circ$

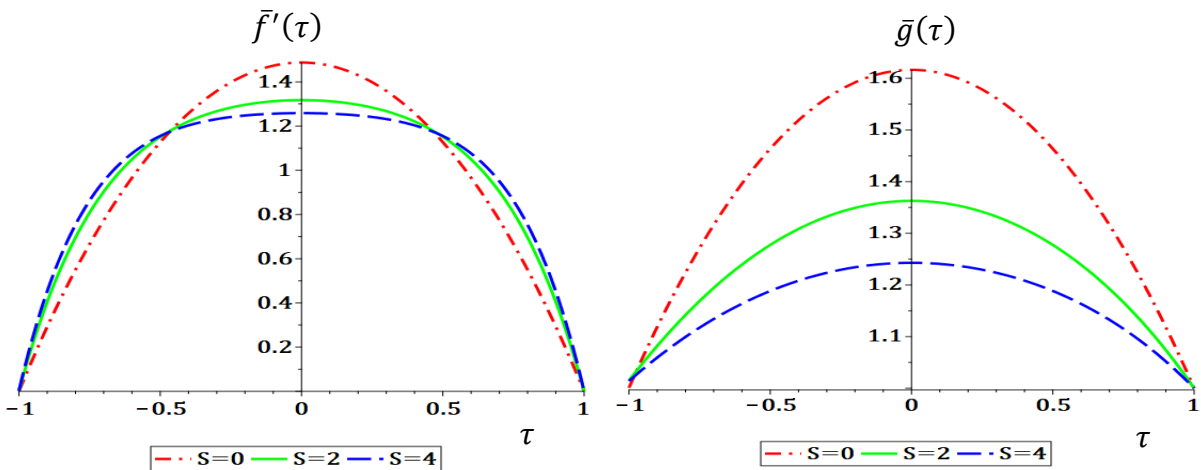
| $w$  | q-HAM          | LTCP-q-HAM     | BVP4C          |
|------|----------------|----------------|----------------|
| 0.02 | 5.656451759679 | 5.819942961723 | 5.506047396726 |
| 0.06 | 6.182353139054 | 6.062674455654 | 5.730750033108 |
| 0.08 | 6.374019472624 | 6.143500460889 | 5.816291998248 |
| 0.10 | 6.530378615155 | 6.190565259759 | 5.887118518898 |
| 0.15 | 6.800421388610 | 6.293706421903 | 6.010969183689 |

The impact of squeezing numbers on velocity and temperature profiles can be seen in Figures (2) and (3), there are two trends for the velocity profile in the interval  $\tau \in [0, 0.5]$ : the flow velocity decreases as the squeezing number absolute value increases. In the other direction, the behavior of the second interval  $\tau \in [0.5, 1]$  is entirely different. It was observed that the velocity rises as the squeeze number grows, starting at a critical value  $\tau_c \cong 0.5$ . Conversely, the temperature profiles. It should be noted that when the squeeze number increases, there is a commensurate drop in temperature and boundary-layer thickness. This is due to  $S$  dependency on  $\varphi$  for the nanoparticles of  $Au$  and  $Fe_3O_4$ . It is evident from Figures (4) and (5) that there are two trends for  $\bar{f}'(\tau)$ . When the Hartmann number falls within the interval,  $\tau \in [0, 0.5]$ , a decrease in flow velocity is observed. Conversely, in the interval,  $\tau \in [0.5, 1]$ , the flow exhibits distinct behaviors, where an increase in the Hartmann number leads to a rise in velocity from a critical point  $\tau_c \cong 0.5$ . Additionally, the values of  $\bar{g}(\tau)$  illustrate the influence of the Hartmann number and other studied parameters on temperature profiles. Notably, an increase in this parameter results in a temperature drop, consequently reducing the boundary layer thickness. As the inclination angle of the applied magnetic field increases, it leads to flattening velocity and temperature in the center of the channel, as shown in Figures (6) and (7). The temperature and flow velocity take low values compared to the results obtained for the vertical magnetic field. Figures (8) and (9) illustrate how the solid volume concentration of  $Au$ , and  $Fe_3O_4$

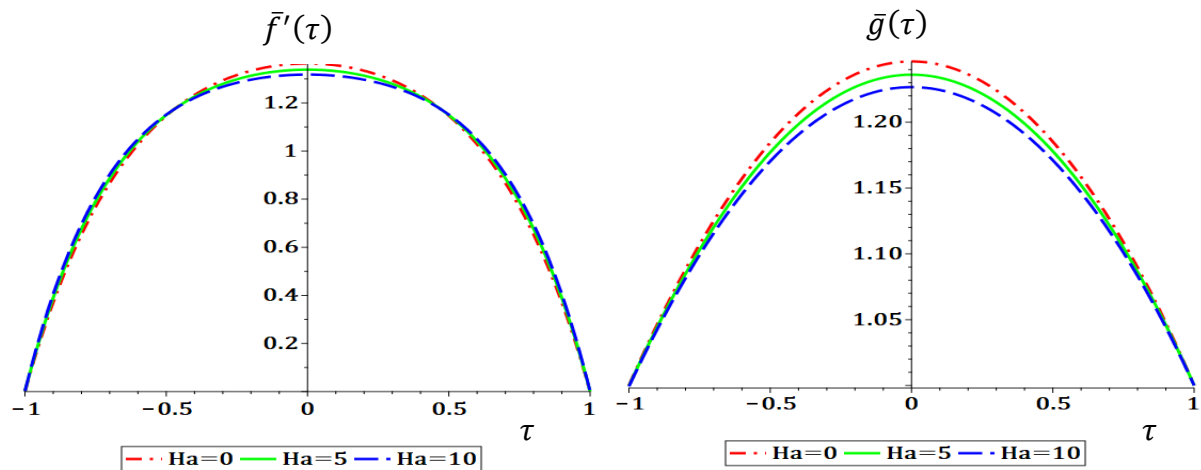
nanoparticles affect the temperature distribution and flow velocity, respectively. It is discovered that the two figures of nanoparticle effect are identical, with high values of velocity occurring as solid volume concentration and temperature increase. Finally, the impact of  $\varepsilon, \delta, E_c, P_r$  for various nanoparticle materials can be visualized.



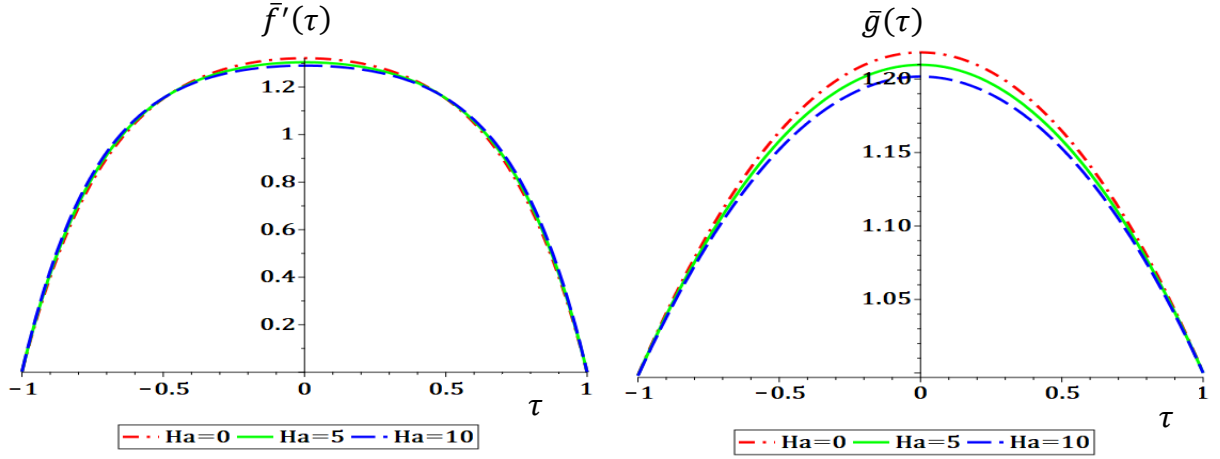
**Fig. 2.**  $\bar{f}'(\tau)$  and  $\bar{g}(\tau)$  for  $Fe_3O_4$ ,  $\varphi = 45^\circ$ ,  $E_c = 0.5$ ,  $\varepsilon = 0.5$ ,  $P_r = 7$ ,  $Ha = 1$ ,  $w = 0.05$ ,  $\delta = 0.1$



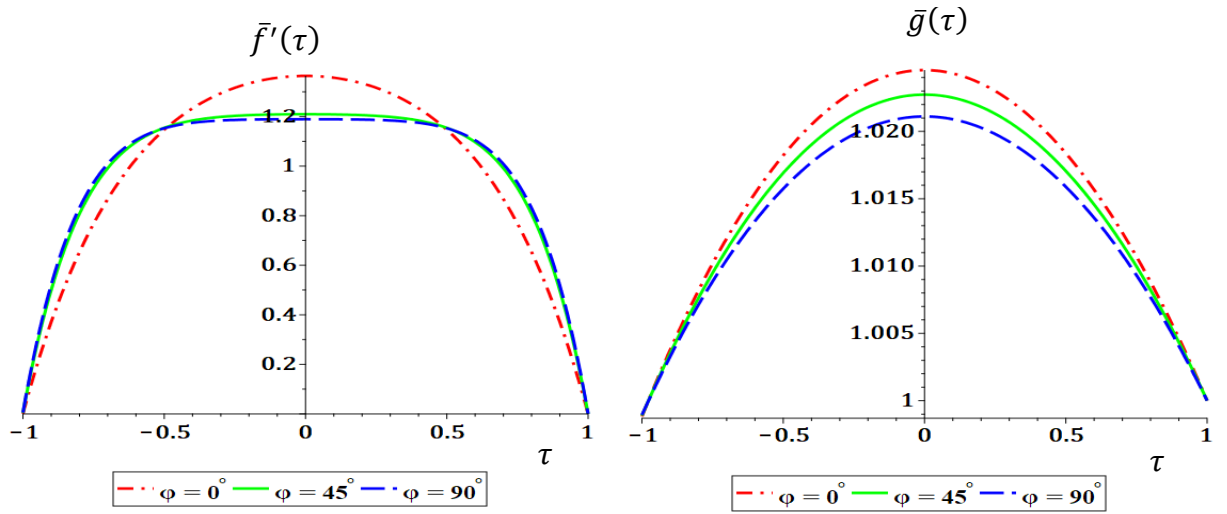
**Fig. 3.**  $\bar{f}'(\tau)$  and  $\bar{g}(\tau)$  for  $Au$ ,  $\varphi = 45^\circ$ ,  $E_c = 0.5$ ,  $\varepsilon = 0.5$ ,  $P_r = 7$ ,  $Ha = 1$ ,  $w = 0.05$ ,  $\delta = 0.1$



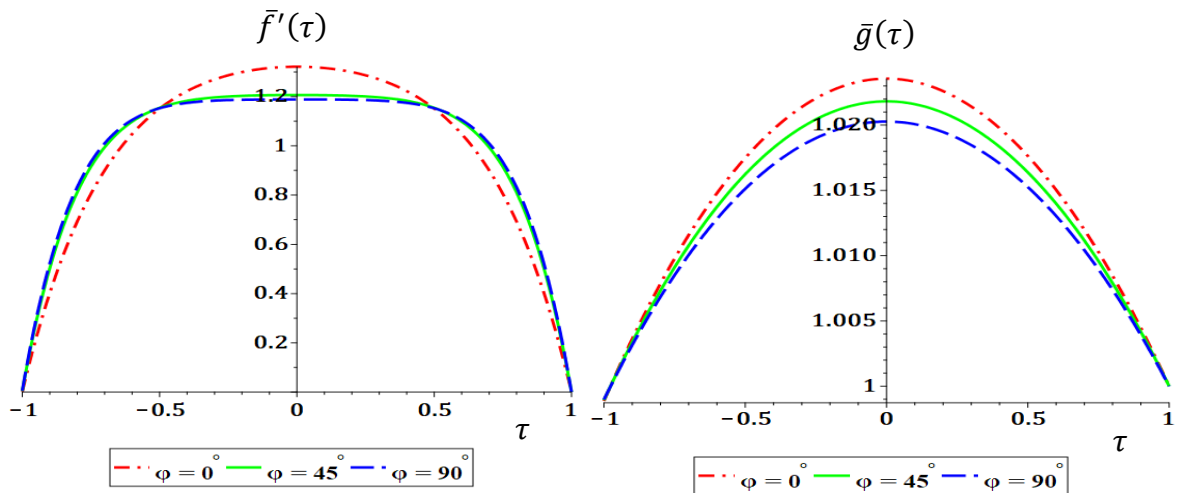
**Fig. 4.**  $\bar{f}'(\tau)$  and  $\bar{g}(\tau)$  for  $Fe_3O_4$ ,  $\varphi = 45^\circ$ ,  $E_c = 0.5$ ,  $\varepsilon = 0.5$ ,  $P_r = 7$ ,  $S = 2$ ,  $w = 0.05$ ,  $\delta = 0.1$



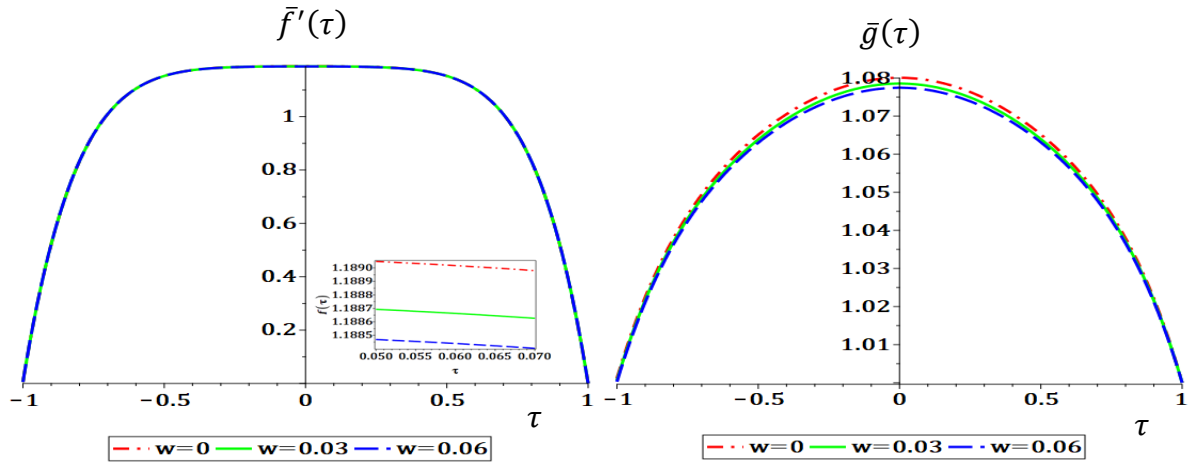
**Fig. 5.**  $\bar{f}'(\tau)$  and  $\bar{g}(\tau)$  for  $Au$ ,  $\varphi = 45^\circ$ ,  $E_c = 0.5$ ,  $\varepsilon = 0.5$ ,  $P_r = 7$ ,  $S = 2$ ,  $w = 0.05$ ,  $\delta = 0.1$



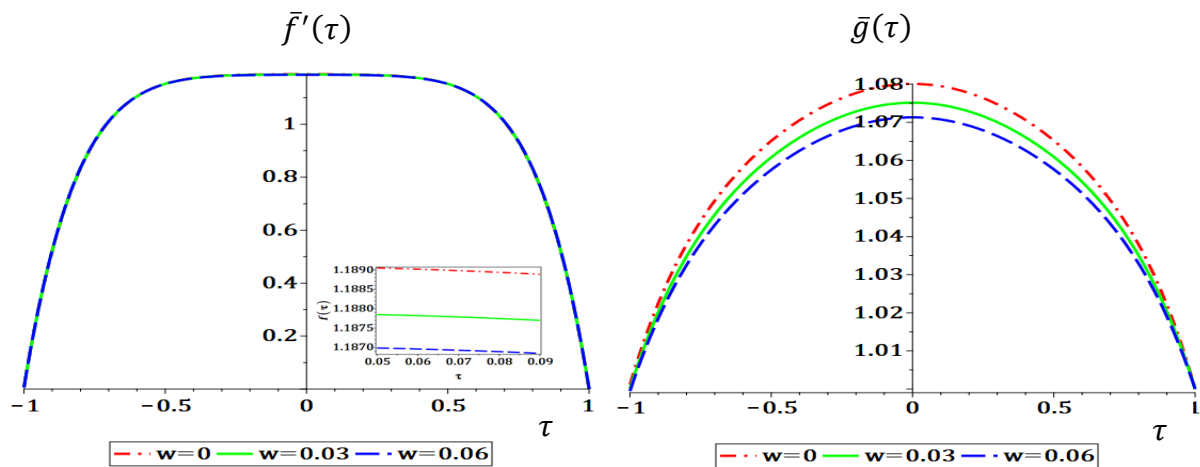
**Fig. 6.**  $\bar{f}'(\tau)$  and  $\bar{g}(\tau)$  for  $Fe_3O_4$ ,  $S = 2$ ,  $E_c = 0.5$ ,  $\varepsilon = 0.5$ ,  $P_r = 7$ ,  $Ha = 100$ ,  $w = 0.05$ ,  $\delta = 0.1$



**Fig. 7.**  $\bar{f}'(\tau)$  and  $\bar{g}(\tau)$  for  $Au$ ,  $S = 2$ ,  $E_c = 0.5$ ,  $\varepsilon = 0.5$ ,  $P_r = 7$ ,  $Ha = 100$ ,  $w = 0.05$ ,  $\delta = 0.1$



**Fig. 8.**  $\bar{f}'(\tau)$  and  $\bar{g}(\tau)$  for  $Fe_3O_4$ ,  $S = 3$ ,  $E_c = 0.5$ ,  $\varepsilon = 0.4$ ,  $P_r = 7$ ,  $Ha = 200$ ,  $\varphi = 45^\circ$ ,  $\delta = 0.1$



**Fig. 9.**  $\bar{f}'(\tau)$  and  $\bar{g}(\tau)$  for  $Au$ ,  $S = 3$ ,  $E_c = 0.5$ ,  $\varepsilon = 0.4$ ,  $P_r = 7$ ,  $Ha = 200$ ,  $\varphi = 45^\circ$ ,  $\delta = 0.1$

The measure errors for q-HAM and LTCP-q-HAM can be indicated in **Table (12)**, this Table demonstrates that the error measures of and for LTCP-q-HAM are accurate from q-HAM.

**Table 12.** The comparison of the errors of  $\bar{f}(\tau)$  and  $\bar{g}(\tau)$  between q-HAM, LTCP-q-HAM, when  $n = 2$ ,  $h = -1.9$ ,  $Ha = 10$ ,  $S = 0$ ,  $\varphi = 1^\circ$ ,  $w = 0.02$ ,  $P_r = 0.1$ ,  $E_c = 0.01$ ,  $\delta = 0.1$ ,  $\varepsilon = 0.01$ ,  $\hat{\rho}_s = 19300$

| Error      | $\bar{f}(\tau)$        |                        | $\bar{g}(\tau)$        |                        |
|------------|------------------------|------------------------|------------------------|------------------------|
|            | q-HAM                  | LTCP-qHAM              | q-HAM                  | LTCP-qHAM              |
| $L_1$      | $2.01 \times 10^{-14}$ | $9.57 \times 10^{-17}$ | $5.95 \times 10^{-12}$ | $8.84 \times 10^{-14}$ |
| $L_2$      | $1.42 \times 10^{-7}$  | $9.79 \times 10^{-9}$  | $2.44 \times 10^{-6}$  | $2.97 \times 10^{-7}$  |
| $L_\infty$ | $2.11 \times 10^{-7}$  | $1.45 \times 10^{-8}$  | $3.17 \times 10^{-6}$  | $3.54 \times 10^{-7}$  |

## 7. Conclusions

In this work, the effects of inclined magnetic fields and variable thermal conductivity on heat transfer squeezing flow of unstable nanofluids have been investigated. The relevant nonlinear partial differential equations were transformed into a set of ordinary differential equations that have been solved analytically and numerically. The general conclusion can be submitted as follows:

- The nanoparticles  $Au$  and  $Fe_3O_4$ , when dispersed in the base fluid,  $H_2O$ , exhibit a similar influence on the obtained solutions. The presence of the base fluid significantly enhances the heat transfer properties, contributing to improved thermal performance.
- The incorporation of an inclined magnetic field, coupled with variable thermal conductivity, leads to a reduction in heat transfer efficiency.
- The convolution theory has a big impact on optimizing the results of solutions that solve the nanofluid problems.
- For certain values of the governing physical parameters, the solutions obtained using the proposed technique, are more accurate than those obtained using the q-homotopy analysis method.
- The comparison results of velocity and temperature profiles with the results of previous studies found that the results obtained from the proposed technique have a high level of agreement with the numerical solution.

## References

- [1] S. S. Reddy, K. Govardhan, G. Narendar, and S. Misra, "Investigating the effect of an inclined magnetic field on heat and mass transmission in turbulent squeeze flow of UCM fluid between parallel plates," *Arch. Thermodyn.*, pp. 169–178, 2024. DOI: <https://doi.org/10.24425/ather.2024.152006>.
- [2] A. Dawar and H. Khan, "Analytical and numerical solutions of squeezing nanofluid flow between parallel plates: A comparison of IRPSM and HPM," *Adv. Mech. Eng.*, vol. 16, no. 12, 2024. DOI: <https://doi.org/10.1177/16878132241304600>.
- [3] G. Domairry and M. Hatami, "Squeezing Cu–water nanofluid flow analysis between parallel plates by DTM-Padé Method," *J. Mol. Liq.*, vol. 193, pp. 37–44, 2014. DOI: <http://dx.doi.org/10.1016/j.molliq.2013.12.034>.
- [4] A. Nouar, A. Dib, M. Kezzar, M. R. Sari, and M. R. Eid, "Numerical treatment of squeezing unsteady nanofluid flow using an optimized stochastic algorithm," *Z. Naturforsch. A*, vol. 76, no. 10, pp. 933–946, 2021. DOI: <https://doi.org/10.1515/zna-2021-0163>.
- [5] S. Panda, L. A. AL-Essa, R. Baithalu, S. R. Mishra, and A. Saeed, "Velocity slip impact on the squeezing flow of water–CNTs and kerosene–CNTs nanofluids over a Riga surface," *J. Taibah Univ. Sci.*, vol. 18, no. 1, 2024. DOI: <https://doi.org/10.1080/16583655.2024.2438324>.
- [6] A. Rehman, Z. Salleh, and M. Zeb, "Approximate analytical study of unsteady flow and heat transfer analysis of carbon nanotubes nanofluid over stretching sheet: Article 3," *Journal of Mathematical Sciences and Informatics*, vol. 1, no. 1, pp. 25–38, 2021. DOI: <http://dx.doi.org/10.46754/jmsi.2021.12.003>.
- [7] H. Shool, A. K. Raheem, and A. M. Al-Jaberi, "New Analytical and Numerical Solution for Squeezing Flow between Parallel Plates under Slip," *Iraqi Journal of Science*, 2024. DOI: <http://dx.doi.org/10.24996/ij.2024.65.3.34>.
- [8] N. Hedayati and A. Ramiar, "Investigation of two-phase unsteady nanofluid flow and heat transfer between moving parallel plates in the presence of the magnetic field using GM," *Challenges in Nano and Micro Scale Science and Technology*, vol. 4, pp. 52–58, 2016. DOI: <http://dx.doi.org/10.7508/tpnms.2016.02.006>.
- [9] T. Hayat, T. Muhammad, A. Qayyum, A. Alsaedi, and M. Mustafa, "On squeezing flow of nanofluid in the presence of magnetic field effects," *J. Mol. Liq.*, vol. 213, pp. 179–185, 2016. DOI: <http://dx.doi.org/10.1016/j.molliq.2015.11.003>.
- [10] A. G. Madaki, R. Roslan, M. S. Rusiman, and C. S. K. Raju, "Analytical and numerical solutions of squeezing unsteady Cu and  $TiO_2$ -nanofluid flow in the presence of thermal

- radiation and heat generation/absorption,” *Alex. Eng. J.*, vol. 57, no. 2, pp. 1033–1040, 2018 DOI: <http://dx.doi.org/10.1016/j.aej.2017.02.011>.
- [11] M. Kezzar, I. Tabet, and N. Nafir, “Semi-analytical and numerical solutions for nonlinear problem of unsteady squeezing ferrofluid flow between stretchable/shrinkable walls under external magnetic field and thermal radiation using differential transformation method,” *Journal of Nanofluids*, vol. 8, pp. 297–307, 2019. DOI: <http://dx.doi.org/10.1166/jon.2019.1599>.
- [12] M. R. Eid, “Chemical reaction effect on MHD boundary-layer flow of two-phase nanofluid model over an exponentially stretching sheet with a heat generation,” *J. Mol. Liq.*, vol. 220, pp. 718–725, 2016. DOI: <http://dx.doi.org/10.1016/j.molliq.2016.05.005>.
- [13] M. R. Eid and K. L. Mahny, “Unsteady MHD heat and mass transfer of a non-Newtonian nanofluid flow of a two-phase model over a permeable stretching wall with heat generation/absorption,” *Adv. Powder Technol.*, vol. 28, no. 11, pp. 3063–3073, 2017. DOI: <https://doi.org/10.1016/j.appt.2017.09.021>.
- [14] K. Al-Farhany, M. F. Al-Dawody, D. A. Hamzah, and N. H. Hamza, “Numerical study of nanofluid natural convection in a partially heated tall enclosure,” *IOP Conf. Ser. Mater. Sci. Eng.*, vol. 928, no. 2, p. 022137, 2020. DOI: <https://doi.org/10.1088/1757-899X/928/2/022137>.
- [15] Y. S. Daniel, Z. A. Aziz, Z. Ismail, and F. Salah, “Thermal radiation on unsteady electrically conducting MHD flow of nanofluid over stretching sheet with chemical reaction,” *J. King Saud Univ. Sci.*, vol. 31, no. 4, pp. 804–812, 2019. DOI: <https://doi.org/10.1016/j.jksus.2017.10.002>.
- [16] S. Hussain, A. Shah, A. Ullah, and F. Haq, “The q-homotopy analysis method for a solution of the Cahn–Hilliard equation in the presence of advection and reaction terms,” *J. Taibah Univ. Sci.*, vol. 16, no. 1, pp. 813–819, 2022. DOI: <https://doi.org/10.1080/16583655.2022.2119746>.
- [17] B. Wu and Y. Qian, “Padé Approximation Based on Orthogonal Polynomial,” in *Proceedings of 2016 International Conference on Modeling, Simulation and Optimization Technologies and Applications (MSOTA2016)*, 2016. DOI: <https://doi.org/10.2991/msota-16.2016.54>.
- [18] H. A. A. Namoos and A. M. Jasim, “The analytical improvement of q-homotopy analysis method for magneto-hydrodynamic (MHD) Jeffrey–Hamel nanofluid flow,” *Adv. Res. Fluid Mech. Therm. Sci.*, vol. 119, no. 2, pp. 32–55, 2024. DOI: <https://doi.org/10.37934/arfm.119.2.3255>.
- [19] A. Jasim and A.-S. J. A.-S. Majeed, “New analytical solution formula for heat transfer in unsteady two-dimensional squeezing flow of a Casson fluid between parallel circular plates,” *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences*, vol. 64, no. 2, pp. 219–243, 2019. Available at: [https://semarakilmu.com.my/journals/index.php/fluid\\_mechanics\\_thermal\\_sciences/article/view/3650](https://semarakilmu.com.my/journals/index.php/fluid_mechanics_thermal_sciences/article/view/3650).



## تأثير نظرية الألتواء في انتقال الحرارة لتدفق الموائع النانوية غير المستقرة مع وجود مجال مغناطيسي مائل

حيدر عبد الرزاق ناموس\*، عبير مجيد جاسم.

قسم الرياضيات، كلية العلوم، جامعة البصرة، البصرة، العراق.

| معلومات البحث                                                                                                                                                                                           | الملخص                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| الاستلام 5 كانون الثاني 2025<br>المراجعة 7 شباط 2025<br>القبول 15 شباط 2025<br>النشر 31 حزيران 2025                                                                                                     | تبحث الدراسة في انتقال حرارة الموائع النانوية غير المستقرة في تدفق مضغوط بين لوحين متوازيين باستخدام الماء كسائل أساسي مع جزيئات الذهب ( $Au$ ) والمغنيتيت ( $Fe_3O_4$ ) النانوية. تم استخدام منهج تحليلي جديد (LCP-q-HAM) يجمع بين طريقة تحليل (q-HAM) وتحويل لابلاس ونظرية الألتواء وتقريب بادي لحل المعادلات التفاضلية غير الخطية التي تتضمن التوصيل الحراري وتأثيرات المجال المغناطيسي. تتم مقارنة الحلول التحليلية والعديدية باستخدام BVP4C من خلال الجداول والرسوم البيانية وتحليل توزيعات درجة الحرارة والسرعة لمعطيات مختلفة مثل جزء حجم الجسيمات النانوية ورقم هارتمان ورقم الضغط وزاوية المجال المغناطيسي والموصلية. وتؤكد النتائج فعالية الطريقة التحليلية. |
| <b>الكلمات المفتاحية</b><br>طريقة التحليل كيو هو موتوبي،<br>تحويل لابلاس، نظرية<br>الألتواء، تقريب بادي،<br>التوصيل الحراري المتغير،<br>الانضغاط غير المستقر، زاوية<br>ميل المجال المغناطيسي<br>المطبق. |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |

**Citation:** H. A. A. Namooos, A. M. Jasim, J. Basrah Res. (Sci.) 51(1), 68 (2025).  
DOI: <https://doi.org/10.56714/bjrs.51.1.6>

\*Corresponding author email : haedirnamoos@gmail.com

