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Modified Non-Polynomial Fractional Spline for Solving a System of Differential Equations

Paywast J. Hasan[1], Faraidun K. Hamasalh[2,*]

[1]University of Sulaimani, Department of Mathematics, College of Education, Sulaimani, Iraq.
[2]Sulaimani Polytechnic University, Bakrajo Technical Institute, Sulaimani, Iraq.

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ABSTRACT

This work presents a fractional non-polynomial spline approach for solving a system of differential equations based on physics, engineering, and optimal control. The approach that produces a system of differential equations and more clearly illustrates the variety of the non-polynomial interpolation method demonstrates the diversity of methods within the Caputo derivative frame. The proposed methodology shows the possibility of obtaining numerical results for a system of differential equations in a variety of scientific fields.

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1. Introduction:

Mathematical models involving systems of differential equations play a fundamental role in many areas of science, engineering, and physics. These equations are widely used to model and simulate physical processes, as well as to test and verify heat equations [1]. Among these, boundary value problems (BVPs) with various boundary conditions serve as powerful tools for describing realistic scenarios, and a system of different orders is commonly applied in real-world contexts based on observed data [2].

Furthermore, researchers have developed various numerical algorithms to obtain accurate and efficient solutions. For example, Geng and Cui [3] investigated a second-order nonlinear system using the reproducing kernel method, while Khalid et al. [4] and Choudhary et al. [5] used numerical techniques to solve linear and nonlinear differential equations. Lu [6] extended a variational iteration method for approximating solutions, and Laplace-based homotopy analysis methods have been modified by Bataineh et al. [7] and Karwan et al. [8] to extend the error estimations by a spline function.

Non-polynomial spline interpolation has been extensively applied in multiple domains, including classic polynomial splines in solving systems of BVPs, fractional differential equations, and fluid mechanics. These functions have been extended to be highly developed in approximating results. For

*Corresponding author email: faraidun.hamasalh@spu.edu.iq

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instance, Caglar, Aniley, and Heilat et al. [9,10, 11, and 15] used B-spline scaling functions to address second-order systems, and Faraidun et al. [18 and 19] applied non-polynomial splines based on fractional-order methods to fractional differential problems. Ragula et al.[13] extended singularly perturbed differential equations with mixed shifts using a non-polynomial spline, while Emadifar et al. [14] proposed a scheme using extended non-polynomial splines for sub-diffusion equations cases. Additionally, Tesfaye and his team [22] successfully applied an extended spline scheme to solve singularly perturbed parabolic problems.

2. Basic Definitions

In this section, we present the basic definitions that are used, and we will discuss the fundamental concepts associated with the non-polynomial spline and fractional derivatives. It will cover specific definitions of fractional derivatives, such as:

2.1 Definition: Gamma function [16]

The Gamma function $\Gamma(.)$, is defined by

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx$$

2.2 Definition: Caputo derivatives [16]

The Caputo fractional derivative operator ${}^{c}_{\alpha}D^{\alpha}_{t}$ of order, $\alpha \in \mathbb{R}^{+}$of a function, $u \in C^{m}_{-1}[a,b]$ and $\alpha \in (m-1, m], m \in \mathbb{N}$ is defined as:

Thus, for $= m$, $m \in \mathbb{N}$, and $u \epsilon C^{m}[a,b]$, we have ${}^{C}_{a}D^{0}_{a}u(t) = u(t)$and ${}^{C}_{a}D^{m}_{a}u(t) = u^{(m)}(t)$.

$${}^{C}D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt.$$

2.3 Definition Riemann-Liouville fractional derivatives [16]:

The Riemann-Liouville fractional derivative operator ${}^{RL}_{\alpha}D^{\alpha}_{t}$ of order $\alpha \geq 0, m = \lceil \alpha \rceil$, ceiling function, and $u \in C^{m}_{-1}[a,b]$, is normally defined as:

$${}^{RL}_{\alpha}D^{\alpha}_{t} = D^{m}_{t} a J^{m-\alpha}_{t} u(t)$$

where , if $\alpha = m, m \in \mathbb{N}$, and $u \in C^{m}[a,b]$, we have ${}^{RL}_{a}D^{m}_{a}u(t) = u^{(m)}(t)$.

3. Modified Non-Polynomial Spline Interpolation

In non-polynomial spline interpolation, the derivative at a given point is used in constructing the interpolating polynomial. To compute the derivatives at the selected nodes, we use an approximation based on finite differences, as well as a Taylor series expansion. This provides an efficient way to handle fractional derivatives without requiring complex symbolic calculations. We can set a mesh of points of an equally spaced partition of an interval [a, b], dividing it into N equally spaced: $a=x_0<x_1<x_2<\dots\dots\dots<x_n=b$, $\theta = kh$, h=(b-a)/N, as follows:

$$p_j(x) = a\cos k(x-x_j) + be^{k(x-x_j)} + c(x-x_j)^{1/2} + d \quad, j=0, 1, 2, \dots, \; N_j \tag{1}$$

The interpolation conditions: $p_j(x_{j+1}) = s_{j+1}$, $p_j(xj) = s_j$, $p_j{}^{(2)}(x_j) = M_j$, and $p_j{}^{(2)}(x_{j+1}) = M_{j+1}$.

Theorem 3.1: (Uniqueness of Non-polynomial Interpolation with Fractional Order)

Let $\{x_i\}_{i=0}^{n}$ be distinct, uniformly spaced nodes in the interval$[a,b]$, and let $s(x)$ be a function such that it's fractional derivative $D^2 p(x)$ and $s(x)$ exists at each x_i and continuous on $[a,b]$, then there exists a unique interpolating function $p_j(x)$.

Proof: Using the non-polynomial spline interpolation in equation (1), with the four conditions, after some simplifications by taking the derivatives and interpolating at each x_i, we can obtain the following coefficients:

$$a = \frac{k^2(s_j - s_{j+1}) - M_j - 4\theta^2 M_{j+1} + e^{\theta} M_j + 4M_j\theta^2 e^{\theta}}{k^2(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})},$$

$$b = \frac{k^2(s_j - s_{j+1}) - \cos\theta M_j + M_j + 4\theta^2\cos\theta M_j - 4M_{j+1}\theta^2}{k^2(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})},$$

$$c = \frac{-4h^{3/2}[(2\cos\theta e^{\theta} - \cos\theta - e^{\theta})M_j + (2 - e^{\theta} - \cos\theta)M_{j+1} + (\cos\theta - e^{\theta})k^2 s_j + (e^{\theta} - \cos\theta)k^2 s_{j+1}}{(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})},$$

$$d = \frac{2k^2 s_{j+1} + (4\theta^2\cos\theta - 4\theta^2 e^{\theta} - \cos\theta - e^{\theta})k^2 s_j + (\cos\theta - e^{\theta} - 4\theta^2\cos\theta - 4\theta^2 e^{\theta})M_j + 8\theta^2 M_{j+1}}{k^2(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})}.$$

Suppose two interplants $p_j(x)$ and $p_j^{(2)}(x)$ satisfy the conditions, as found in the above coefficients, since $p(x)$ is a linear combination of basis interpolation functions at all nodes, then $p_j(x)$ is a unique polynomial similar to [18, 19].

3.1 Iterative of the Non-Polynomial Fractional Spline Function

This modification presents an iterative formulation based on the given formula of Theorem 3.1 for solving a system of differential equations. The basic ideas and concepts are adapted from foundational methods introduced in [20] and further developed using fractional polynomial techniques with Caputo derivatives [21, 22].

Let the fractional spline range polynomials $p_i(x)$ form the basis, where j=0, 1, …, N and $\theta = kh$, at the knots, apply the half derivative continuity conditions of spline functions to acquire the following iterative formulas:

$$p_j^{(1/2)}(x) = p_j^{(1/2)}(x).$$

And evaluating all the coefficients of equation (1), we obtain

$$p_j(x) = \frac{k^2(s_j - s_{j+1}) - M_j - 4\theta^2 M_{j+1} + e^{\theta} M_j + 4M_j\theta^2 e^{\theta}}{k^2(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})}\cos k(x - x_j) +$$

$$\frac{k^2(s_j - s_{j+1}) - \cos\theta M_j + M_j + 4\theta^2\cos\theta M_j - 4M_{j+1}\theta^2}{k^2(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})} e^{k(x - x_j)} -$$

$$\frac{4h^{3/2}[(2\cos\theta e^{\theta} - \cos\theta - e^{\theta})M_j + (2 - e^{\theta} - \cos\theta)M_{j+1} + (\cos\theta - e^{\theta})k^2 s_j + (e^{\theta} - \cos\theta)k^2 s_{j+1}}{(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})}(x - x_j)^{1/2} +$$

$$\frac{2k^2 s_{j+1} + (4\theta^2\cos\theta - 4\theta^2 e^{\theta} - \cos\theta - e^{\theta})k^2 s_j + (\cos\theta - e^{\theta} - 4\theta^2\cos\theta - 4\theta^2 e^{\theta})M_j + 8\theta^2 M_{j+1}}{k^2(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})},$$

Compute the fractional derivatives based on $p(x)$, to form each x_i, we obtain the iterative formula:

$$s_{j+1} - 2s_j + s_{j-1} = -\frac{1}{k^2}\left[\alpha M_{j+1} + \beta M_j + \gamma M_{j-1}\right] \qquad (2)$$

j=1,…, N-1.

Where,

$$\alpha = \frac{(1+\sqrt{2})4\theta^2 + (4 - 2e^{\theta} - 2\cos\theta)\sqrt{2}\sqrt{\pi}\theta^{3/2}}{1 + \sqrt{2} + 2\sqrt{2}\sqrt{\pi}\theta^{3/2}e^{\theta} - 2\sqrt{2}\sqrt{\pi}\theta^{3/2}\cos\theta},$$

$$\beta = \frac{1 - e^{\theta} - (e^{\theta} + 1 + \sqrt{2})4\theta^2 - \sqrt{2} - 4\sqrt{2}\sqrt{\pi}\theta^{3/2} - (4\theta^2 - 1 - 4\sqrt{\pi}\theta^{3/2})\sqrt{2}\cos\theta}{1 + \sqrt{2} + 2\sqrt{2}\sqrt{\pi}\theta^{3/2}e^{\theta} - 2\sqrt{2}\sqrt{\pi}\theta^{3/2}\cos\theta},$$

and $$\gamma = \frac{\left(1 + 4\theta^2 + 2\sqrt{2}\sqrt{\pi}\theta^{3/2}\right)e^{\theta} + \sqrt{2} + \left(4\theta^2 - 1 - 4\sqrt{\pi}\theta^{3/2}e^{\theta} + 2\sqrt{\pi}\theta^{3/2}\right)\sqrt{2}\cos\theta - 1}{1 + \sqrt{2} + 2\sqrt{2}\sqrt{\pi}\theta^{3/2}e^{\theta} - 2\sqrt{2}\sqrt{\pi}\theta^{3/2}\cos\theta}.$$

These formulas enable the construction of a compact iterative formula that encapsulates fractional derivatives, consistent with methods from [15, 21, and 22].

4. Convergence Analysis

The convergence analysis reveals that the proposed non-polynomial spline method demonstrates superior accuracy and stability compared to the classical Taylor series, particularly for functions with higher-order derivatives or localized behavior. While the Taylor series relies on polynomial approximations centered around a single point, the spline approach captures the function's behavior over intervals, allowing for faster convergence with fewer terms. This makes the non-polynomial spline especially effective for problems where the Taylor series may diverge or require a high number of terms to achieve comparable precision.

Lemma 4.1: Let $f \in C^{\alpha}([a,b])$ be a function with fractional smoothness $\alpha \in (a,b]$, and let $p_j(x)$ denote the non-polynomial spline-type fractional interplant constructed from values $f(x_i)$ and fractional derivatives $D^{\alpha}f(x_i)$ at nodes $\{x_i\}_{i=0}^{n}$. Then the interpolation error satisfies:

$$\left|f^{(m)}(x) - p^{(m)}{}_j(x)\right| \leq C\omega(h)h^m, \ m=0,\ 1,\ 2.$$

where:

- $h = \max|x_{i+1} - x_i|$ is the maximum spacing between the nodes,
- $\omega(h)$ is the modulus 1of continuity of $D^{\alpha}f$,
- C is a constant depending only on α and the interpolation scheme.

Proof: The modulus of continuity of a function f, denoted by $\omega(f,\delta)$, is defined as:

$$\omega(f,\delta) = \sup_{\substack{x,y\in[a,b]\\ |x-y|\leq\delta}} |f(x) - f(y)|.$$

For fractional derivatives of f, define:

$$\omega_{\alpha}(f,\delta) = \sup_{\substack{x,y\in[a,b]\\ |x-y|\leq\delta}} |\mathcal{D}^{\alpha}f(x) - \mathcal{D}^{\alpha}f(y)|, \quad \alpha \in \{0,1,2\}.$$

We start by expanding $f(x)$ at each node x_i using the Taylor series with fractional remainder:

$$f(x) = f(x_i) + \frac{2}{\sqrt{\pi}}(x - x_i)^{\frac{1}{2}} f^{\left(\frac{1}{2}\right)}(x_i) + (x - x_i)f'(x_i) + \frac{4}{3\sqrt{\pi}}(x - x_i)^{\frac{3}{2}} f^{\left(\frac{3}{2}\right)}(x_i) + R_2(x, x_i),$$

The non-polynomial spline is

$$p_j(x) = \frac{k^2(s_j - s_{j+1}) - M_j - 4\theta^2 M_{j+1} + e^{\theta}M_j + 4M_j\theta^2 e^{\theta}}{k^2(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})} \cos k(x - x_j) +$$

$$\frac{k^2(s_j - s_{j+1}) - \cos\theta M_j + M_j + 4\theta^2\cos\theta M_j - 4M_{j+1}\theta^2}{k^2(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})} e^{k(x - x_j)} -$$

$$\frac{4h^{3/2}[(2\cos\theta e^{\theta} - \cos\theta - e^{\theta})M_j + (2 - e^{\theta} - \cos\theta)M_{j+1} + (\cos\theta - e^{\theta})k^2 s_j + (e^{\theta} - \cos\theta)k^2 s_{j+1}}{(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})}(x - x_j)^{1/2} +$$

$$\frac{2k^2 s_{j+1} + (4\theta^2\cos\theta - 4\theta^2 e^{\theta} - \cos\theta - e^{\theta})k^2 s_j + (\cos\theta - e^{\theta} - 4\theta^2\cos\theta - 4\theta^2 e^{\theta})M_j + 8\theta^2 M_{j+1}}{k^2(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})} \tag{3}$$

Since $|x - x_i| \leq h$ and $\omega(|x - x_i|) \leq \omega(h)$, and letting $x = x_{j+1}$, to obtain:

$$\left|p_j(x) - f(x)\right| = \left|\frac{1}{k^2(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})}[(k^2(s_j - s_{j+1}) - M_j - 4\theta^2 M_{j+1} + e^{\theta}M_j +\right.$$

$$4M_j\theta^2 e^{\theta})\cos\theta + (k^2(s_j - s_{j+1}) - \cos\theta M_j + M_j + 4\theta^2\cos\theta M_j - 4M_{j+1}\theta^2)e^{\theta} - ((2\cos\theta e^{\theta} -$$

$$\cos\theta - e^{\theta})M_j + \left(2 - e^{\theta} - \cos\theta\right)M_{j+1} + \left(\cos\theta - e^{\theta}\right)k^2 s_j + \left(e^{\theta} - \cos\theta\right)k^2 s_{j+1}\Big) 4\theta^2 +$$

$$2k^2 s_{j+1} + \left(4\theta^2\cos\theta - 4\theta^2 e^{\theta} - \cos\theta - e^{\theta}\right)k^2 s_j + \left(\cos\theta - e^{\theta} - 4\theta^2\cos\theta - 4\theta^2 e^{\theta}\right)M_j +$$

$$8\theta^2 M_{j+1}] - [y_j + \frac{2h^{1/2}}{\sqrt{\pi}} y_j{}^{1/2} + h y_j{}' + \frac{4h^{3/2}}{3\sqrt{\pi}} y_j{}^{3/2} + \frac{h^2}{2} y_j{}''(\theta 1)\ ]\Big|,$$

Substitute $s_j = p_j$,$s_{j+1} = p_{j+1}$,$M_j = p_j''$, $M_{j+1} = p_{j+1}''$ and
$\alpha_1 = \frac{1}{k^2(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})}$, then after simplification, we get:

$$\left|p_j(x) - f(x)\right| = \Big|\alpha_1\big(-8\theta^2 e^{\theta}\cos\theta p_j + \left(k^2 e^{\theta} - k^2\cos\theta - 4\theta^2 k^2 e^{\theta} + 4\theta^2 k^2\cos\theta +\right.$$

$$\left.2k^2\right)p_{j+1}\big) - [p_j + \frac{2h^{1/2}}{\sqrt{\pi}} p_j{}^{1/2} + h p_j{}' + \frac{4h^{3/2}}{3\sqrt{\pi}} p_j{}^{3/2} + \frac{h^2}{2} p_j{}''(\theta 1)\ ]\Big| \tag{4}$$

let $\alpha_2 = -8\theta^2 e^{\theta}\cos\theta$, $\alpha_3 = k^2 e^{\theta} - k^2\cos\theta - 4\theta^2 k^2 e^{\theta} + 4\theta^2 k^2\cos\theta + 2k^2$

Therefore: $\left|f(x) - p_j(x)\right| = |E(x)| \le C\omega(h)h^2,\ h = \max|x - x_i|$
$\left|\sqrt{x - x_i} \cdot \mathcal{D}^{\frac{1}{2}} f(x_i)\right| \le \sqrt{h} \cdot \| \mathcal{D}^{\frac{1}{2}} f \|_{\infty}$ and $\left|(x - x_i)^{\frac{3}{2}} \cdot \mathcal{D}^{\frac{3}{2}} f(x_i)\right| \le h^{\frac{3}{2}} \cdot \| \mathcal{D}^{\frac{3}{2}} f \|_{\infty}$.

and simplifying with constants $C_1 = (\alpha_1\alpha_2 + \alpha_1\alpha_3 - \alpha_4), C_2 = (\alpha_1\alpha_3 - \alpha_4), C_3 = (\alpha_1\alpha_3 - \alpha_4)$ $C_4 = (\alpha_1\alpha_3 - \alpha_4)$ from equation (4), independent h, and can be written as:
$\omega\left(p_j(x), \delta\right) \le (C_1 h^{\frac{3}{2}} + C_2 h^2 + C_3 h + C_3 h^{\frac{1}{2}})\ \omega(f, \delta)$,

We get,
$\left|f(x) - p_j(x)\right| = |E(x)| \le Ch^2\omega(h)$, where $Ch^2 = \mathrm{Max}(C_1 h^{\frac{3}{2}} + C_2 h^2 + C_3 h + C_3 h^{\frac{1}{2}})$.

Similarly, we can find an error bound for the derivatives as follows:
$\alpha_1 = \frac{1}{k(-\cos\theta + 2 - e^{\theta} + 4\theta^2\cos\theta - 4\theta^2 e^{\theta})}$ We expanded all p_{j+1} and p_{j+1}'', after simplification, we get:

$$\left|p_j{}'(x) - f'(x)\right| = \Big|\alpha_1\big[(k^2\cos\theta - k^2\sin\theta)p_j + (k^2\sin\theta - k^2\cos\theta)p_{j+1} + \big(\sin\theta - e^{\theta}\sin\theta -$$

$$4e^{\theta}\theta^2\sin\theta - e^{\theta}\cos\theta + e^{\theta} - 4e^{2\theta}\theta^2 - 4he^{\theta}\cos\theta + 2h\cos\theta + 2he^{\theta}\big)p_j'' + \big(4\theta^2\sin\theta - 4e^{\theta}\theta^2 -$$

$$\cos\theta - e^{\theta} + 2\big)p_{j+1}''\big] - \left(p_j' + \frac{2h^{1/2}}{\sqrt{\pi}} p_j{}^{3/2} + h p_j''(\theta 1)\right)\Big|,$$

Let $\alpha_2 = k^2\cos\theta - k^2\sin\theta$, $\alpha_3 = k^2\sin\theta - k^2\cos\theta$, $\alpha_4 = \sin\theta - e^{\theta}\sin\theta - 4e^{\theta}\theta^2\sin\theta - e^{\theta}\cos\theta + e^{\theta} - 4e^{2\theta}\theta^2 - 4he^{\theta}\cos\theta + 2h\cos\theta + 2he^{\theta}$ and
$\alpha_5 = 4\theta^2\sin\theta - 4e^{\theta}\theta^2 - \cos\theta - e^{\theta} + 2$,

$$\left|p_j{}'(x) - f'(x)\right| = \Big|\alpha_1\big[\alpha_2 p_j + \alpha_3 p_{j+1} + \alpha_4 p_j'' + \alpha_5 p_{j+1}''\big] - (p_j' + \frac{2h^{1/2}}{\sqrt{\pi}} p_j{}^{3/2} + h p_j''(\theta 1))\Big|,$$

We expand $p_{j+1}$and p_{j+1}'' Then, after simplification, we get:

$$\left|p_j{}'(x) - y_j{}'(x)\right| = \Big|(\alpha_1\alpha_2 + \alpha_1\alpha_3)p_j + (\alpha_1\alpha_3)\frac{2h^{1/2}}{\sqrt{\pi}} p_j{}^{1/2} + (\alpha_1\alpha_3 h - 1)p_j' + p_j{}'' +$$

$$\left(\alpha_1\alpha_3\frac{4h^{3/2}}{3\sqrt{\pi}} - \frac{2h^{1/2}}{\sqrt{\pi}}\right)p_j{}^{3/2} + h p_j''(\theta 1) - \frac{h^2}{2} p_j{}''(\theta 2)\Big|,$$ in particular, we obtain:

$$\left|p_j{}'(x) - f'(x)\right| = |E(x)| \le Ch^3\omega(h),$$

where $Ch^2 = \mathrm{Max}((\alpha_1\alpha_2 + \alpha_1\alpha_3)p_j + (\alpha_1\alpha_3)\frac{2h^{1/2}}{\sqrt{\pi}}p_j^{1/2} + (\alpha_1\alpha_3 h - 1)p_j' + \left(\alpha_1\alpha_3\frac{4h^{3/2}}{3\sqrt{\pi}} - \frac{2h^{1/2}}{\sqrt{\pi}}\right)p_j^{3/2})$. And finally,

$$\left|p_j''(x) - f''(x)\right| = \left|\alpha_1\left[\left(4\theta^2\cos\theta - 4\theta^2 e^{\theta} - e^{\theta} - \cos\theta\right)p_{j+1}'' + 4\theta^2 e^{2\theta}p_j''\right] - [p_j'' + \frac{2h^{1/2}}{\sqrt{\pi}}p_j^{5/2} + hp_j'''(\theta_1)]\right|,$$

let $\alpha_2 = \left(4\theta^2\cos\theta - 4\theta^2 e^{\theta} - e^{\theta} - \cos\theta\right)$ and $\alpha_3 = 4\theta^2 e^{2\theta}$,

After expanded p_{j+1}'', and simplify the last step, we get:

$$\left|p_j''(x) - f''(x)\right| = \left|(\alpha_1\alpha_2 + \alpha_1\alpha_3 - \alpha_4)p_j'' + (\alpha_1\alpha_2 - \alpha_4)\frac{2h^{1/2}}{\sqrt{\pi}}p_j^{5/2} + \alpha_1\alpha_2 hp_j'''(\theta_2) + \alpha_4 hp_j'''(\theta_1)\right|$$

and simplifying with constants $C_1 = (\alpha_1\alpha_2 + \alpha_1\alpha_3 - \alpha_4), C_2 = (\alpha_1\alpha_3 - \alpha_4), C_3 = \alpha_1\alpha_2 + \alpha_4$ from equation (4), independent h, and can be written as:

$$\omega\left(p_j(x), \delta\right) \le (C_1 + C_2 h^{1/2} + C_3 h)\ \omega(f, \delta) \le Hh^4\omega(f, \delta),$$

where $Hh^4 = \mathrm{Max}((C_1 + C_2 h^{1/2} + C_3 h)$.

5. Numerical Discussion

The numerical approximations show the usefulness of our developed non-polynomial spline; we have solved three problems of a system of second-order differential equations. All computations for these problems were carried out using the Maple software. The test cases confirm high accuracy and stability, particularly in problems involving memory effects and anomalous diffusion. As shown in Figures (5.1-5.3), the value of absolute errors decreases significantly, approaching the exact solutions, and is comparable with results in [12] and [17]. These findings validate fractional non-polynomial spline interpolation as a reliable and practical tool for numerical solutions of systems of differential equations.

Problem 5.1: [17] Consider the system of differential equations

$$u'' + xu + xv = 2$$

$$v'' + 2xv + xu = -2$$

u(0) = u(1) = v(0) = v(1) = 0, 0<x<1, the exact solutions are $u(x) = x^2 - x$, and $v(x) = x - x^2$.

Table 5.1: Numerical Solution and Absolute Error of $v_j(x)$ to Problem 5.1.

x	$v_j(x)$exact	$v_j(x)$approximation	Absolute error
0	0.0000	0.0000	0.0000
0.05	0.0475	0.04802403	0.0005
0.1	0.0900	0.09118100	0.001181
0.15	0.1257	0.12944150	0.00374150
0.2	0.1600	0.16276986	0.00276986
0.25	0.1875	0.19112592	0.00362592
0.3	0.2100	0.21446690	0.00446690
0.35	0.2275	0.23274916	0.00524916
0.4	0.2400	0.24593008	0.00593008
0.45	0.2475	0.25396985	0.00646985
0.5	0.2500	0.25683331	0.00683331
0.55	0.2475	0.25449176	0.00699176
0.6	0.2400	0.24692477	0.00692477
0.65	0.2275	0.23412203	0.00662203

0.7	0.2100	0.21608517	0.00608517
0.75	0.1875	0.19282954	0.00532954
0.8	0.1600	0.16438610	0.00438610
0.85	0.1257	0.13080316	0.00510316
0.9	0.0900	0.09214830	0.00214830
0.95	0.0475	0.04851009	0.00101009
1	0.0000	0.0000	0.0000

Table 5.2: Numerical Solution and Absolute Error of $u_j(x)$ to Problem 5.1

x	$u_j(x)$exact	$u_j(x)$approximation	Absolute error
0	0.0000	0.0000	0.0000
0.05	-0.0475	-0.04742093	0.00012093
0.1	-0.0900	-0.08985019	0.00025019
0.15	-0.1257	-0.12728777	0.00158777
0.2	-0.1600	-0.15973367	0.00033367
0.25	-0.1875	-0.18718790	0.00048790
0.3	-0.2100	-0.20965045	0.00045045
0.35	-0.2275	-0.22712132	0.00042132
0.4	-0.2400	-0.23960051	0.00040051
0.45	-0.2475	-0.24708803	0.00058803
0.5	-0.2500	-0.24958387	0.00058387
0.55	-0.2475	-0.24708803	0.00058803
0.6	-0.2400	-0.23960051	0.00040051
0.65	-0.2275	-0.22712132	0.00042132
0.7	-0.2100	-0.20965043	0.00045043
0.75	-0.1875	-0.18718790	0.00048790
0.8	-0.1600	-0.15973367	0.00033367
0.85	-0.1257	-0.12728777	0.00158777
0.9	-0.0900	-0.08985019	0.00025019
0.95	-0.0475	-0.04742093	0.00012093
1	0.0000	0.0000	0.0000

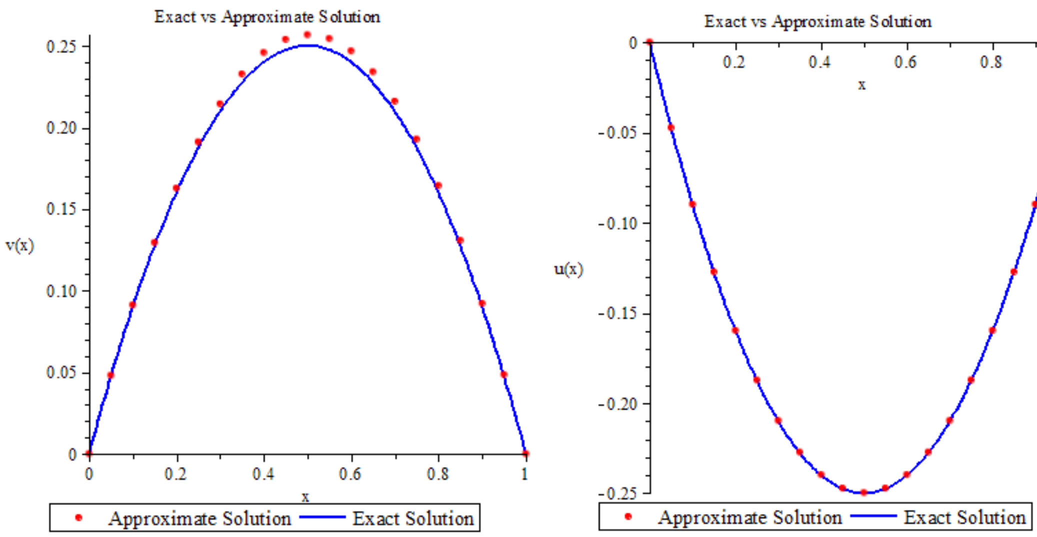

Fig. 5.1: The Graph between the approximate solution and exact solution of problem 5.1

Problem 5.2: [17] Consider the following differential equations

$$u'' + (2x-1)\,u' + \cos(\pi x)\,v' = -\pi^2 \sin(\pi x) + (2x-1)(\pi + 1)\cos(\pi x)$$

$$v'' + xu = 2 + x\sin(\pi x)$$

$u(0) = u(1) = v(0) = v(1) = 0$, $0<x<1$, the exact solutions are $u(x) = \sin(\pi x)$, and $v(x) = x^2 - x$.

Solution:

Table 5.3: Numerical Solution and Absolute Error of $u_j(x)$ to Problem 5.2

x	u_j(x)exact	u_j(x)approximation	Absolute error
0	0.0000	0.0000	0.0000
0.05	0.15635581	0.15652021	0.00016440
0.1	0.30886552	0.30520389	0.00366162
0.15	0.45377762	0.44283463	0.01094299
0.2	0.58752752	0.56646789	0.02105963
0.25	0.70682518	0.67350347	0.03332170
0.3	0.89873606	0.76174943	0.04698662
0.35	0.89075331	0.82947603	0.06127727
0.4	0.95085946	0.87545829	0.07540116
0.45	0.98757597	0.89900608	0.08856989
0.5	0.99999968	0.89998113	0.10001855
0.55	0.98782499	0.87880043	0.10902455
0.6	0.95135137	0.83642597	0.11492540
0.65	0.89147602	0.77434105	0.11713497
0.7	0.80967178	0.69451385	0.11515793
0.75	0.70795090	0.59934896	0.10860194
0.8	0.58881556	0.49162825	0.09718730
0.85	0.45519628	0.37444245	0.08075382
0.9	0.31037990	0.25111513	0.05926477
0.95	0.15792867	0.12512090	0.03280777
1	0.0000	0.00159265	0.00159265

Table 5.4: Numerical Solution and Absolute Error of $v_j(x)$ to Problem 5.2.

x	v_j(x)exact	v(x)approximation	Absolute error
0	0.0000	0.0000	0.0000
0.05	-0.0475	-0.04742093	0.00012093
0.1	-0.0900	-0.08985019	0.00025019
0.15	-0.1257	-0.12728777	0.00158777
0.2	-0.1600	-0.15973367	0.00033367
0.25	-0.1875	-0.18718790	0.00048790
0.3	-0.2100	-0.20965045	0.00045045
0.35	-0.2275	-0.22712132	0.00042132
0.4	-0.2400	-0.23960051	0.00040051
0.45	-0.2475	-0.24708803	0.00058803
0.5	-0.2500	-0.24958387	0.00058387
0.55	-0.2475	-0.24708803	0.00058803
0.6	-0.2400	-0.23960051	0.00040051
0.65	-0.2275	-0.22712132	0.00042132
0.7	-0.2100	-0.20965043	0.00045043
0.75	-0.1875	-0.18718790	0.00048790
0.8	-0.1600	-0.15973367	0.00033367
0.85	-0.1257	-0.12728777	0.00158777
0.9	-0.0900	-0.08985019	0.00025019
0.95	-0.0475	-0.04742093	0.00012093
1	0.0000	0.0000	0.0000

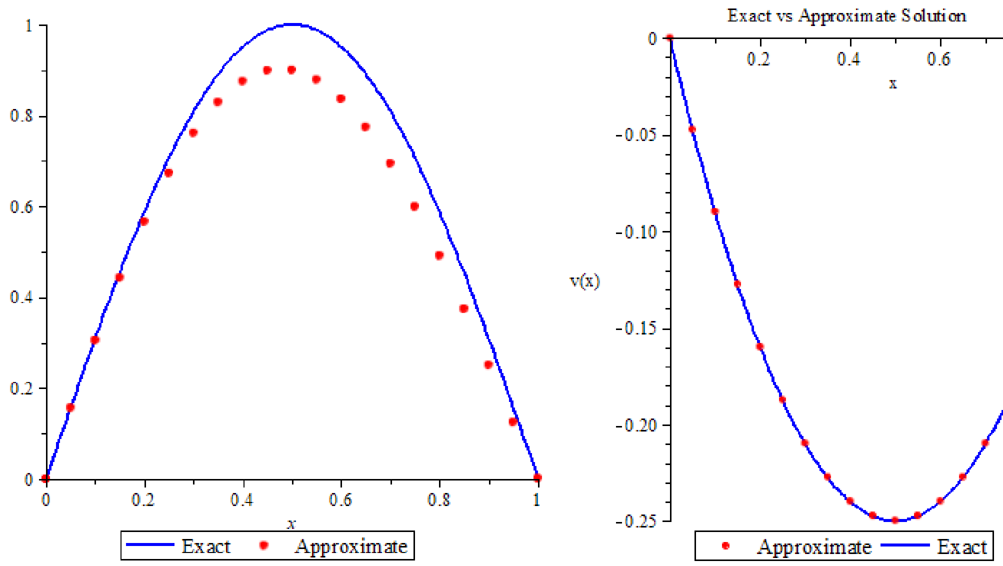

Fig. 5.2: The graph between the approximate solution and exact solution of problem 5.2.

Problem 5.3 [12]: Consider the following non-linear differential equations

$$u'' + xv + xu^2 = f_1(x)$$

$$v'' + xu' + v = f_2(x)$$

The exact solution is $u(x) = \sin\pi x$ and $v(x) = x^3 - 3x^2 + 2x$,
where 0<x<1, u(0) = u(1) = 0 and v(0) = v(1) = 0 , $f_1(x) = x\sin^2\pi x - \pi^2\sin\pi x + x^4 - 3x^3 + 2x^2$
And $f_2(x) = \pi\, x\cos\pi x + x^3 - 3x^2 + 8x - 6$

Table 5.5: Numerical Solution and Absolute Error of $u_j(x)$ to Problem 5.3

x	$u_j(x)$exact	$u_j(x)$approximation	Absolute error
0	0.0000000	0.0000000	0.0000000
0.05	0.1563558	0.1595747	0.0032189
0.1	0.30886552	0.3112170	0.0023515
0.15	0.45377762	0.4516400	0.0021376
0.2	0.58752752	0.5778328	0.0096955
0.25	0.70682518	0.6871343	0.0196908
0.3	0.89873606	0.7772993	0.1214367
0.35	0.89075331	0.8465530	0.0525223
0.4	0.95085946	0.8936352	0.0572242
0.45	0.98757597	0.9178308	0.0697451
0.5	0.99999968	0.9189881	0.0810115
0.55	0.98782499	0.8975216	0.0903039
0.6	0.95135137	0.8544028	0.0969485
0.65	0.89147602	0.7911355	0.1003405
0.7	0.80967178	0.7097192	0.0999525
0.75	0.70795090	0.6125998	0.0953510
0.8	0.58881556	0.5026096	0.0862059
0.85	0.45519628	0.3828973	0.0722989
0.9	0.31037990	0.2568510	0.0535289
0.95	0.15792867	0.128014	0.0298146
1	0.0000	0.0015926	0.0015926

Table 5.6: Numerical Solution and Absolute Error of $v_j(x)$ to Problem 5.3

x	v_j exact	$v_j$approximate	Absolute error
0	0.000000	0.000000	0.000000
0.05	0.092625	0.092733	0.000108
0.1	0.171000	0.170777	0.000223
0.15	0.235875	0.234954	0.000919
0.2	0.288000	0.286085	0.001915
0.25	0.328125	0.324990	0.005633
0.3	0.357000	0.352492	0.004508
0.35	0.375375	0.369411	0.005964
0.4	0.384000	0.376568	0.007432
0.45	0.383625	0.374786	0.008839
0.5	0.375000	0.364884	0.010116
0.55	0.358875	0.347685	0.011190
0.6	0.336000	0.324009	0.011991
0.65	0.307125	0.294678	0.012451
0.7	0.273000	0.260514	0.012486
0.75	0.234375	0.222336	0.012039
0.8	0.192900	0.180967	0.011933
0.85	0.146625	0.137227	0.009398
0.9	0.099000	0.091939	0.007061
0.95	0.049875	0.045922	0.003953
1	0.000000	0.000000	0.000000

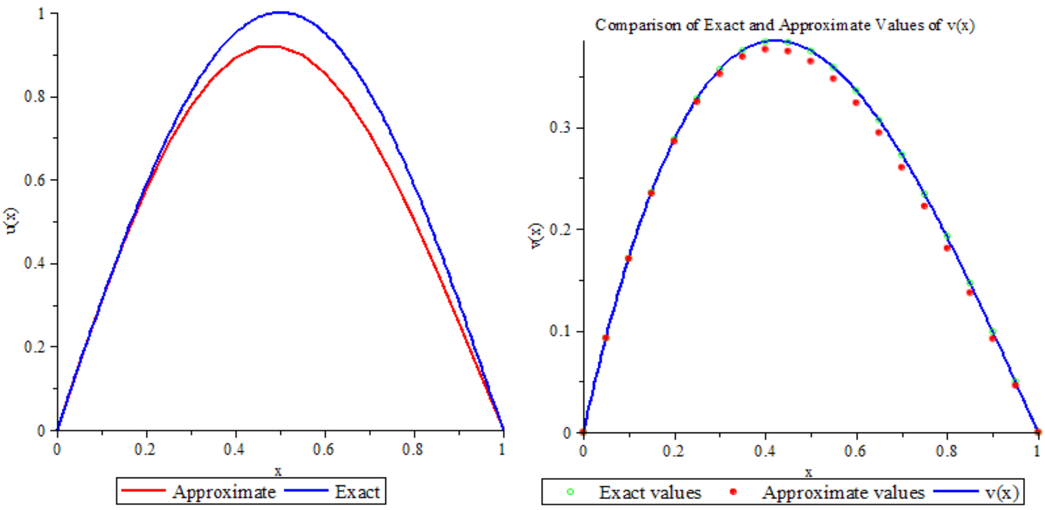

Fig. 5.3: The graph between the approximate solution and exact solution of problem 5.3

6. Conclusion

This paper presents an effective numerical technique for solving a system of differential equations using non-polynomial spline interpolation in fractional order, which is introduced together with the use of the Taylor series expansion, which is based on the Caputo derivative formula, and the solution curve is spatially interpolated using the functions using the Maple software. We obtain the maximum errors between the exact solution and the numerical approach shown in the tables (5.1-5.5) at the uniform points x belonging to [0, 1] concerning the step size h. It offers sufficient precision and is innovative. Additionally, three numerical cases have been examined utilizing the non-polynomial spline approach, and graphs show the accuracy and practicality of the approach. The theoretical findings are validated by the numerical outcomes of the non-polynomial spline interpolation method.

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سبلاين الكسرية غير متعدد الحدود المعدل لحل نظام المعادلات التفاضلية

بيوست جبار حسن[1]، فريدون قادر حمه صالح[2]

[1]جامعة السليمانية، قسم الرياضيات، كلية التربية، السليمانية، العراق
[2]جامعة السليمانية التقنييه، معهد بكرجو التقني، السليمانية، العراق

معلومات البحث

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المراجعة 18 اب 2025
القبول 25 اب 2025
النشر 31 كانون أول 2025

الكلمات المفتاحية

التقريب السبلاين، المشتقة الكسرية لكابوتو، تحليل التقارب، شروط الاستمرارية

الملخص

يستخدم هذا العمل طريقة سبلاين غير متعددة الحدود لحل نظام المعادلات التفاضلية من الرتبة الثانية القائمة على الفيزياء ، الهندسة والتحكم الأمثل. الطريقة التي تنتج نظامًا من المعادلات التفاضلية وتوضح بشكل أكثر وضوحا تنوع طريقة الاستيفاء غير المتعددة الحدود التي تظهر تنوع الطرق داخل إطار مشتقة كابوتو. منهجية هذا العمل تظهراحتمالية الحصول على نتائج عددية لنظام من المعادلات التفاضلية في مجموعة مجالات علمية متنوعة.

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*Corresponding author email: faraidun.hamasalh@spu.edu.iq

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