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Numerical Solutions of One-Dimensional Space-Fractional Diffusion Equation Using Least-Squares-Petrov-Galerkin Approach

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ABSTRACT

In this work, we proposed a novel method for solving one-dimensional space fractional diffusion equations (SFDE) based on combining the least-squares method with Petrov-Galerkin approach, utilizing orthogonal polynomials as basis functions, with the fractional derivative considered in the Caputo-Fabrizio sense. This method is to express the unknown function as a series of orthogonal polynomials that are linearly combined. By using this approach, we can turn the problem into a system of linear algebraic equations that can be solved using MATLAB R2023a for the unknown constants associated with the approximate solution. We provide two examples that illustrate the accuracy of our method and its ability to be applied effectively. The graphs and error tables support the proposed approach's effectiveness and efficiency. The results indicate that the proposed method yields more accurate solutions than others for solving similar problems.

1. Introduction

Fractional calculus is one of the branches of mathematics that has gained considerable popularity and importance because of its attractive applications as a new modeling tool in many areas of scientific and engineering fields that involve memory and non-local properties. It is a generalization of integer-order differentiation and integration. In recent years, this area has received increasing attention, and it has been applied to solve many scientific and life problems, such as in Physics [1], Mathematical biology [2], Image processing [3], and several other subjects. It is often used to characterize the properties of memory and heredity in various materials.

Fractional partial differential equations (FPDEs) are fundamental in many branches of engineering and mathematical areas. They are used in various fields, such as electric transmission, fluid mechanics. Chemical diffusion, ocean circulation, signal processing, solid mechanics, speech signal modelling, wave propagation, etc. [4]. Unlike classical differential equations, non-integer-order differential equations include hereditary characteristics, providing a more realistic depiction of real-world issues.

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Recently, derivatives of arbitrary order have been used to tackle real-world intricate issues. It is well recognized that most fractional differential equations cannot be solved analytically. Consequently, approximation methods have been devised and analyzed to provide approximate solutions for fractional-order differential equations. These comprise the fractional partial differential equations via the Haar wavelet collocation method [5], the fourth-order fractional integro-differential equations using the Legendre-collocation scheme [6], also included are the Fitzhugh–Nagumo equation using the homotopy perturbation technique coupled with the modified Sumudu transform method [7]. Furthermore, covered are the fractional Navier–Stokes equation using the Laplace-Adomian decomposition method [8], the time-fractional-order Helmholtz equations using the modified residual power series method [9], the fractional Fisher's equation with the optimal quintic B-spline approach [10], the time-fractional Zakharov-Kuznetsov equations using Chebyshev spectral method [11], and many other numerical techniques.

In 2015 [12], Caputo and Fabrizio introduced a novel operator known as the Caputo-Fabrizio fractional derivative (CFFD), which is distinguished by its lack of a singular kernel. This operator offers the advantage of providing a more accurate representation of certain phenomena that are often criticized in well-known fractional definitions, such as those by Riemann-Liouville, Caputo, and Riesz, which are limited by their reliance on a singular kernel [13-15]. Traditional local theories often deal with various materials and structures of different sizes. These can be managed more effectively by using non-singular kernels [16,17].

A key advantage of the CFFD method is its ability to apply to a broader range of materials, not just viscoelastic ones, due to its single-kernel representation. This paper utilizes the space-fractional diffusion equation (SFDE), which plays a crucial role in modeling many scientific phenomena, particularly in fields like mathematics and engineering. Many studies have investigated both integer and non-integer order space-fractional partial differential equations. For example, different numerical methods have been used to solve the space-fractional order diffusion equation, including the Chebyshev finite difference method [18], the Tau approach [19], the Galerkin method [20], a Chebyshev pseudo-spectral scheme [21], the spline method [22], and the Legendre-spectral-collocation method [23], additionally, the spline method was employed to solve one and two-dimensional SFDE by Soori and Aminataei in [24]. Safdari et al. in [25] presented a numerical method using compact finite difference and the Chebyshev collocation method to obtain a semi-discrete in-time derivative to approximate the SFDE. Tuan et al. [26] proposed a finite difference scheme combined with a Chebyshev polynomial collocation method of second kind to solve SFDE in the Caputo sense. In [27], Issa et al. introduced a numerical approach using shift Gegenbauer polynomials to solve the SFDE. Nasrudin et al. [28] suggested using shifted two-variable Legendre polynomials and the Ritz method to solve SFDE in the Caputo sense. Hajinezhad and Soheili in [29] developed a finite difference scheme for solving SFDE involving the Caputo fractional derivative. Ayalew et al. proposed the Laguerre spectral collocation method and finite difference scheme for the solution of SFDE [30].

This study focuses on an approximate solution to the following one-dimensional space-fractional diffusion equation (1D-SFDE); it is obtained by replacing the second-order space derivative in the diffusion equation with CFFD of order $1 < \alpha \leq 2$ as [31] was declared:

$$\frac{\partial u(x,t)}{\partial t} = {}_0^{CF}D_x^{\alpha} u(x,t) + f(x,t), \tag{1}$$

with given the initial condition

$$u(x,t) = \psi(x), \quad 0 \leq x \leq L, \quad . \tag{2}$$

and boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = g(t), \quad 0 \leq t \leq T, \tag{3}$$

where $f(x,t)$ is the source term, the symbols, $\psi(x)$ is a given smooth function, T is the final time, and ${}_0^{CF}D_x^{\alpha}$ denote the Caputo-Fabrizio space fractional derivative of order α.

The motivation for this study is to address the significant challenges that researchers encounter in modeling and solving fractional problems and to develop more accurate and efficient numerical methods. This requires accurate and efficient models, one of which is fractional derivatives, which imposes significant complexity in mathematical modeling and in implementing numerical solutions. Moreover, many traditional methods depend on the use of a large number of points to achieve acceptable accuracy, which leads to significant computational costs and poor numerical efficiency, especially for large-scale problems. Therefore, this research aims to present an innovative numerical methodology that is used for the first time to solve the fractional equation (1). The new method combines the least-square method with the Petrov-Galerkin method (LSPGM), using the Caputo-Fabrizio concept and orthogonal polynomials as test and trial functions. This approach provides highly accurate results requiring fewer points compared to traditional methods. This paper presents a significant contribution to the study of fractional differential equations by introducing a robust and efficient numerical method for solving SFDE. The rest of the paper is structured as follows: Section 2 reviews some basic definitions of fractional calculus. Section 3 will present the algorithm of the LSPGM for solving 1D-SFDE. In section 4, some numerical examples are executed, where their results are compared with the numerical solutions of the other methods. Finally, the conclusion is drawn in Section 5.

2. Fundamental definitions

This section focuses on some fundamental concepts and properties of fractional calculus that may be utilized in our research.

Definition 2.1: [32] Let $\alpha \geq 0, n = [\alpha] + 1$ (i.e., the smallest integer greater than or equal to α), the Riemann-Liouville fractional derivative operators of order α for a function f has the following definition:

$$ {}^{RL}_{a}D^{\alpha}_{t} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \left(\dfrac{d}{dt}\right)^{n} \displaystyle\int_{a}^{t} f(\tau)\,(t-\tau)^{(n-\alpha-1)} d\tau, & n-1 \leq \alpha < n \\ f(t), & \alpha = 0 \end{cases} $$

Also, let $\alpha > 0$, then the operator. ${}^{C}_{0}D^{\alpha}_{t}$ Defined by:

$$ {}^{C}_{0}D^{\alpha}_{t} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_{0}^{t} f^{(n)}(\tau)\,(t-\tau)^{(n-\alpha-1)} d\tau, & n-1 \leq \alpha < n,\ n \in \mathbb{N} \\ f(t), & n = \alpha \end{cases} $$

It is called the Caputo fractional derivative operator of order α.

Definition 2.2: [12] Let $0 \leq \alpha \leq 1$, and a function $f(t) \in H^{1}(c,d)$. The Caputo-Fabrizio fractional derivative ${}^{CF}_{a}D^{\alpha}_{t} f(t)$ of order α is defined as:

$$ {}^{CF}_{a}D^{\alpha}_{t} f(t) = \begin{cases} \dfrac{M(\alpha)}{1-\alpha} \displaystyle\int_{a}^{t} f'(\tau) \exp\left(\dfrac{-\alpha(t-\tau)}{1-\alpha}\right) d\tau, & 0 \leq \alpha < 1 \\ \dfrac{df(t)}{dt}, & \alpha = 1 \end{cases} $$

where $M(\alpha)$ is the animalization function that fulfils the conditions of $M(0) = M(1) = 1$.

3. The LSPGM algorithm

The new LSPGM, with the aid of the orthogonal polynomials, Laguerre polynomials, and Chebyshev polynomials [33, 34], is employed in this section to examine the approximate solution for the 1D-SFDE given in Equations (1) and (2).
Now, we set

$$ \mathbb{P}_n = span\{\varphi_i : \quad i = 0,1,\dots,n\}, \qquad n \in \mathbb{Z}, $$

$$ V_n = \{u \in \mathbb{P}_n : u(0,t) = 0, u(L,t) = g\}, $$

where $\mathbb{P}_n$ is the space of all polynomials of degree at most n and $\varphi_i(x)$ is the Laguerre polynomials. The novel method's algorithm is summarized in the following steps:

Step 1: Find $u_n \in V_n$ such that Eq. (1) can be written as:

$$(\frac{\partial u_n}{\partial t}, v_n)_w = ({}^{CF}_{0}D^{\alpha}_{x} u_n, v_n)_w + (f, v_n)_w, \tag{4}$$

$\forall\, v_n \in \mathbb{P}_n$, where

$$(\frac{\partial u_n}{\partial t}, v_n)_w = \int_0^1 \frac{\partial u_n(x)}{\partial t} v_n(x) w(x) dx,$$

$$({}^{CF}_{0}D^{\alpha}_{x} u_n, v_n)_w = \int_0^1 {}^{CF}_{a}D^{\alpha}_{x} u_n v_n(x) w(x) dx,$$

$$(f, v_n)_w = \int_0^1 f(x) v_n(x) w(x) dx.$$

Step 2: The proximate solution is

$$u_n(x) = \sum_{i=0}^{n} c_i\big(\varphi_i(x) + \beta_i \varphi_{i+1}(x)\big), \tag{5}$$

where c_i are undermined constant coefficients, β_i are constants chosen as the initial conditions.

Step 3: Substitute Eq. (5) into Eq. (4), and we obtain

$$\sum_{i=0}^{n} \Big(\frac{\partial}{\partial t}(\varphi_i(x) + \beta_i \varphi_{i+1}(x)), \emptyset_j(x)\Big)_w = \sum_{i=0}^{n} c_i ({}^{CF}_{0}D^{\alpha}_{x}(\varphi_i(x) + \beta_i \varphi_{i+1}(x)), \emptyset_j(x))_w + (f, \emptyset_j(x))_w, \tag{6}$$

where $\emptyset_j(x), j = 0,1,\cdots,n$ are a Chebyshev polynomial test function with its weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$.

Step 4: Consequently, the residual equation for Eq. (6) is defined as follows:

$$\begin{aligned} &R(x; c_0, c_1, \dots, c_n) \\ &= \sum_{i=0}^{n} c_i \left(\frac{\partial}{\partial t}\big(\varphi_i(x) + \beta_i \varphi_{i+1}(x)\big), \emptyset_j(x)\right)_w \\ &- \sum_{i=0}^{n} c_i ({}^{CF}_{0}D^{\alpha}_{x}(\varphi_i(x) + \beta_i \varphi_{i+1}(x)), \emptyset_j(x))_w \\ &- \Big(f, \emptyset_j(x)\Big)_w . \end{aligned} \tag{7}$$

Step 5: Let $S(c_0, c_1, \dots, c_n) = \int_0^1 (R(x; c_0, c_1, \dots, c_n))^2\, w_1(x) dx,$ (8)

where $w_1(x)$ denotes the positive weight function, for simplification, will be chosen $w_1(x) = 1$.

Step 6: Therefore, Eq. (8) becomes:

$$S(c_0, c_1, \dots, c_n) = \int_0^1 \left\{ \sum_{i=0}^{n} c_i \left(\frac{\partial}{\partial t}\left(\varphi_i(x) + \beta_i \varphi_{i+1}(x)\right), \emptyset_j(x) \right)_w - \sum_{i=0}^{n} c_i ({}^{CF}_{0}D^{\alpha}_{x}(\varphi_i(x) + \beta_i \varphi_{i+1}(x)), \emptyset_j(x))_w - \left(f, \emptyset_j(x)\right)_w \right\}^2 dx. \quad (9)$$

Step 7: To minimize Eq. (9), the values $c_i, i = 0, \dots, n$, can be obtained by minimizing values of S as:

$$\frac{\partial S}{\partial c_i} = 0, \quad i = 0,1, \dots, n. \quad (10)$$

Step 8: Applying Eq. (10) on Eq. (9) yields

$$\int_0^1 \left\{ \sum_{i=0}^{n} c_i \left(\frac{\partial}{\partial t}(\varphi_i(x) + \beta_i \varphi_{i+1}(x)), \emptyset_j(x)\right)_w - \sum_{i=0}^{n} c_i ({}^{CF}_{0}D^{\alpha}_{x}(\varphi_i(x) + \beta_i\left(\varphi_{i+1}(x)\right)), \emptyset_j(x))_w - \left(f, \emptyset_j(x)\right)_w \right\} dx \times \int_0^1 \left\{ \left(\frac{\partial}{\partial t}\left(\varphi_i(x) + \beta_i \varphi_{i+1}(x)\right), \emptyset_j(x) \right)_w - ({}^{CF}_{0}D^{\alpha}_{x}(\varphi_i(x) + \beta_i \varphi_{i+1}(x)), \emptyset_j(x))_w \right\} dx = 0. \quad (11)$$

Step 9: Finally, a system of $(n+1)$ equations with $(n+1)$ unknown coefficients $c_i, i = 0,1, \dots, n$ are obtained by solving Eq. (11), which can be formed as follows:

$$A = \begin{bmatrix} \int_0^1 R(x; c_0) h_0 dx & \int_0^1 R(x; c_1) h_0 dx & \cdots & \int_0^1 R(x; c_n)\, h_0 dx \\ \int_0^1 R(x; c_0) h_1 dx & \int_0^1 R(x; c_1) h_1 dx & \cdots & \int_0^1 R(x; c_n) h_1 dx \\ \vdots & \vdots & \vdots & \vdots \\ \int_0^1 R(x; c_0) h_n dx & \int_0^1 R(x; c_1) h_n dx & \cdots & \int_0^1 R(x; c_n) h_n dx \end{bmatrix},$$

$$B = \begin{bmatrix} \int_0^1 \left(f, \emptyset_j\right)_w h_0 dx \\ \vdots \\ \int_0^1 \left(f, \emptyset_j\right)_w h_n dx \end{bmatrix},$$

where $h_i = \left(\frac{\partial}{\partial t}(\varphi_i(x) + \beta_i \varphi_{i+1}(x)), \emptyset_j(x)\right)_w - ({}^{CF}_{0}D^{\alpha}_{x}(\varphi_i(x) + \beta_i\left(\varphi_{i+1}(x)\right)), \emptyset_j(x))_w$,

$$R(x;\ c_0, c_1, \dots, c_n) = \sum_{i=0}^{n} c_i \left(\frac{\partial}{\partial t}(\varphi_i(x) + \beta_i \varphi_{i+1}(x)), \emptyset_j(x) \right)_w - \sum_{i=0}^{n} c_i ({}^{CF}_{0}D^{\alpha}_{x}(\varphi_i(x) + \beta_i \varphi_{i+1}(x)), \emptyset_j(x))_w,$$

thus $A\ U = F,$ (12)

where, $U = [c_0, c_1, \cdots, c_n]^T \qquad i = 0,1,..,n\,.$ (13)

Therefore, by solving the system (12) for the unknown coefficients c_i, $i = 0,1,\dots,n$, and the approximate solution of Eq. (1) is obtained.

4. Numerical Examples

In this section, we provide two numerical examples of the 1D-SFDE to showcase the accuracy and efficiency of the LSPGM. Moreover, all calculations were carried out with MATLAB R 2023a software. To illustrate the accuracy of the proposed method, by using the absolute error end L_2error as defined, respectively, as follows

Absolute error $=|u(x) - u_n(x)| \quad 0 \le x \le L,\ n = 1,2,\dots$

$$L_2 = \|u(x) - u_n(x)\|_2$$

where, $u(x)$ is the exact solution and $u_n(x)$ is the approximate solution.

Example 5.1: Cnsider the following 1D-SFDE of the form [35]

$$\frac{\partial u(x,t)}{\partial t} = {}^{CF}_{0}D^{\alpha}_{x}u(x,t) + f(x,t), \quad 1 < \alpha \le 2,$$

Subject to the initial condition

$$u(x,0) = x^{1+\alpha}, \quad 0 \le x \le 1,$$

and the boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = e^{-t}, \qquad 0 \le t \le 1,$$

where $f(x,t) = e^{-t}(-x^{1+\alpha} - \Gamma(2+\alpha)x)$, and the exact solution is $u(x,t) = e^{-t}x^{1+\alpha}$.

The absolute errors and L_2-errors were derived by implementing the proposed approach for various values of n and α, as shown in Table 1. The numerical outcomes of the proposed strategy significantly surpass those of the method delineated in [35]. In terms of solution accuracy, the new method achieved a much lower absolute error with lower n values than the largest n value in the cited source, which achieves an absolute error of 3.59×10^{-6} at $n = 64$. This reveals the efficiency of the proposed method in reducing errors and improving the accuracy of the solution. The absolute error, approximate and exact solutions for $\alpha = 1.5$ are plotted in Fig. 1 and Fig. 2.

Table 1. Absolute error and L_2 error for different values for n by LSPGM for Example 4.1 with different values for α.

α	n	Absolute error	L_2-error
1.1	4	8.3608×10^{-4}	9.4289×10^{-3}
	5	4.1598×10^{-5}	1.3524×10^{-4}
	6	4.4565×10^{-5}	1.3631×10^{-4}
	7	3.2012×10^{-5}	7.1797×10^{-4}
1.5	4	1.7205×10^{-5}	2.7244×10^{-5}
	5	4.7045×10^{-6}	8.4597×10^{-6}
	6	6.1623×10^{-6}	1.4558×10^{-4}
	7	1.2717×10^{-6}	1.9353×10^{-5}
1.9	4	1.5550×10^{-6}	2.4219×10^{-5}
	5	4.4778×10^{-6}	1.1943×10^{-5}
	6	1.0703×10^{-5}	3.04537×10^{-5}
	7	5.9288×10^{-6}	1.0463×10^{-5}

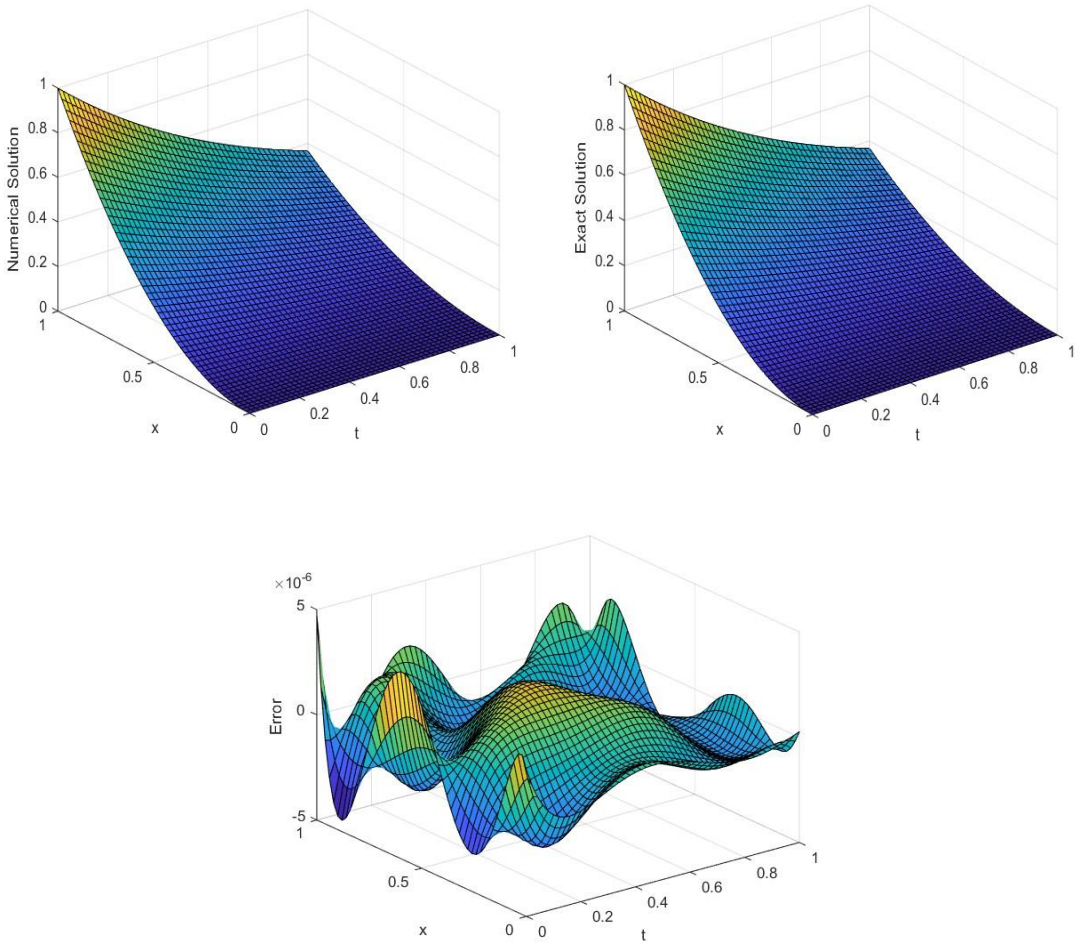

Fig. 1. Plot of the exact solution, approximate solutions, and absolute error for example 4.1 at $\alpha = 1.5$.

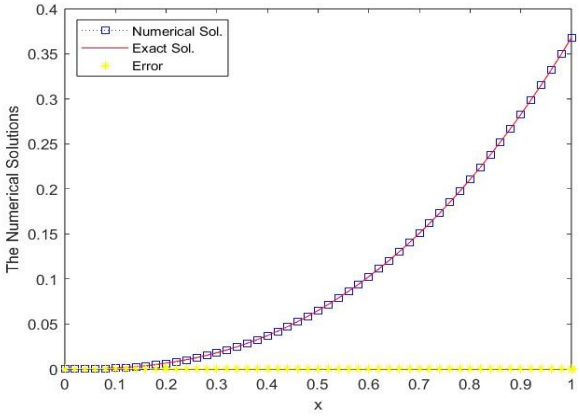

Fig. 2. A comparison of the exact solution, approximate solutions, and absolute error for Example 4.1 with $\alpha = 1.5, 1.1$.

Example 5.2: Consider the following 1D-SFDE of the form [36]:

$$\frac{\partial u(x,t)}{\partial t} + {}_0^{CF}D_x^{\alpha} u(x,t) = f(x,t), \quad 0 < \alpha < 1,$$

Subject to the initial condition

$$u(x,0) = x^2, \quad 0 < x \leq 1,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = t + 1, \quad 0 < t \leq 1,$$

where $f(x,t) = (1.91116 + 1.9116t)x^{1.1} + x^2$, and the exact solution is $u(x,t) = x^2(t+1)$.

Table 2 includes the exact values with the results for different values of n and $\alpha = 0.9$ and compares the absolute errors of the new method and the errors for the same equation in [36]. Our proposed method achieved ideal numerical accuracy by obtaining exact agreement with the analytical solution at all test points x. This results in an impressive zero absolute error across the entire computational domain since the solution of a fractional differential equation usually introduces discretization errors, even with fine grids. The method used in [36] to solve SFDE for $n = 3$ results in a small absolute error between 8.88×10^{-16} and 1.77×10^{-16}. In contrast, our method for $n = 2$ substantially minimized these errors. The comparison results demonstrate the accuracy and efficiency of the proposed method in approximating SFDE. In Table 3, the absolute errors obtained by the proposed method for different values of α with different values of n, the obtained results that could confirm the efficiency of the suggested method. Furthermore, Figs 3 and 4 illustrate the absolute errors with the approximate and exact solutions for the specific case of $\alpha = 0.5$.

Table 2. Numerical results and Absolute error for $\alpha = 0.9$ by using LSPGM for Example 4.2

x	Exact solution	Numerical solution	Absolute error	
			Proposed method n=2	[36] n=3
0.0	0	0	0	8.88178×10^{-16}
0.2	0.0800	0.0800	0	9.71445×10^{-16}
0.4	0.3200	0.3200	0	9.99201×10^{-16}
0.6	0.7200	0.7200	0	1.22125×10^{-15}

0.8	0.2800	0.2800	0	1.11022×10^{-15}
1.0	2.0000	2.0000	0	1.77636×10^{-16}

Table 3. Absolute error for different values of n and α by $LSPGM$ for Example 4.2.

α	n	**Absolute error**
0.5	2	2.5163×10^{-15}
	3	1.8874×10^{-15}
	4	9.2149×10^{-15}
0.25	2	9.6589×10^{-15}
	3	2.5757×10^{-14}
	4	1.0710×10^{-15}
0.9	2	1.5321×10^{-15}
	3	1.3645×10^{-14}
	4	7.8271×10^{-15}

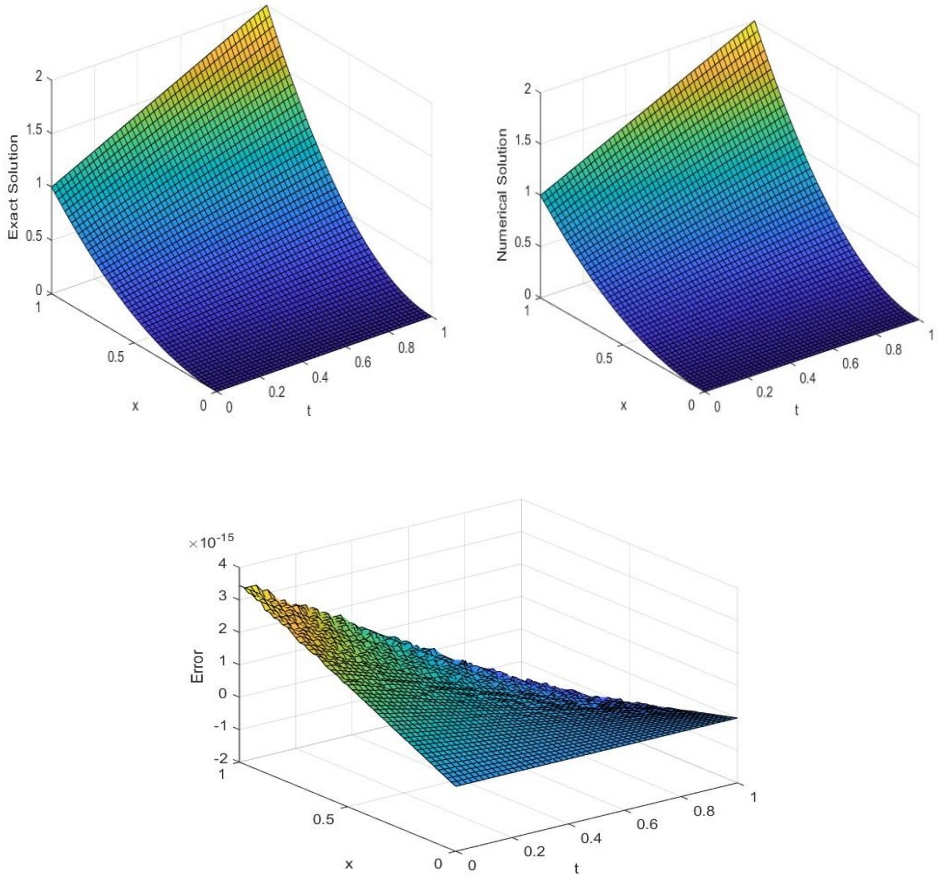

Fig. 3. Plot of the exact solution, approximate solutions, and absolute error for Example 4.2.

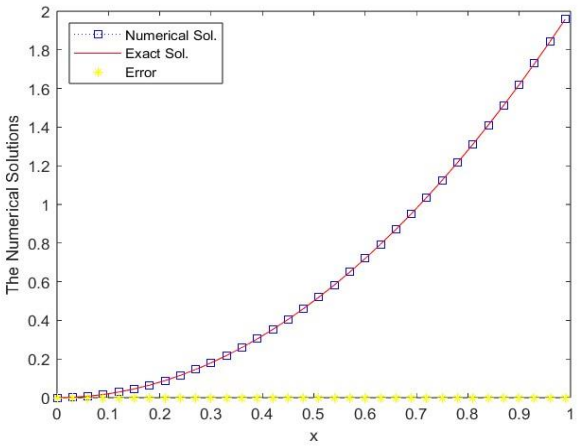

Fig. 4. The exact solution, approximate solutions for Example 4.2 with $\alpha = 0.5$.

5. Conclusion

This study can be described as a novel and efficient numerical approach that consolidates two techniques: the least squares technique and the Petrov-Galerkin method for addressing one-dimensional space-fractional diffusion equations with boundary constraints. Two issues with the current methodology have been addressed. The correctness and efficiency were validated by juxtaposing the numerical results with analytical solutions and findings documented in pertinent literature. Numerical examples testify that the suggested technique yields precise and efficient numerical solutions for space-fractional diffusion issues.

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الحلول العددية لمعادلة الانتشار الكسرية للفضاء احادية البعد بأستخدام طريقة المربعات الصغرى- بيتروف - كَالركن

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معلومات البحث

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الكلمات المفتاحية

معادلة الانتشار الكسرية في الفضاء أحادي البعد، طريقة المربعات الصغرى، طريقة بيتروف-جالركن، مشتق كابوتو فابريزيو الكسري، متعددة الحدود شيبشيف، متعددة الحدود لاگوير.

الملخص

في هذا العمل، اقترحنا طريقة جديدة لحل معادلات الانتشار الكسري في الفضاء أحادي البعد (SFDE) تعتمد على دمج طريقة المربعات الصغرى مع نهج بيتروف-جالركن، باستخدام كثيرات الحدود المتعامدة كدوال أساسية، مع اعتبار المشتق الكسري بمعنى كابوتو-فابريزيو. هذه الطريقة هي نعبرعن الدالة المجهولة كسلسلة من كثيرات الحدود المتعامدة التي يتم دمجها خطيًا. باستخدام هذا النهج، يمكننا تحويل المسألة إلى نظام من المعادلات الجبرية الخطية التي يمكن حلها باستخدام MATLAB R2023a للثوابت المجهولة المرتبطة بالحل التقريبي. نقدم مثالين يوضحان دقة طريقتنا وقدرتها على التطبيق الفعال. الرسوم البيانية وجداول الأخطاء تدعم فعالية وكفاءة النهج المقترح. تشير النتائج إلى أن الطريقة المقترحة تحقق حلولاً أكثر دقة من غيرها في حل المشكلات المماثلة.

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