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Reconstruction of Time-Dependent Source Terms in Fractional Diffusion Equations Using Carleman Estimates and Optimization Algorithms

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A B S T R A C T

This paper investigates the inverse problem of reconstructing time-dependent source terms in time-fractional diffusion equations with Caputo derivatives, which are widely used to model anomalous subdiffusive processes. The objective is to recover the temporal behaviour of unknown source functions from final-time boundary measurements. The methodology integrates both analytical and computational steps. First, a new Carleman estimate is established, ensuring uniqueness and conditional stability of the inverse problem. Next, this theoretical guarantee is used to design a numerical inversion framework. The forward problem is discretized using finite-difference and spectral schemes, and the source term is reconstructed through a regularized Levenberg–Marquardt optimization algorithm. Numerical experiments are then performed on synthetic and perturbed datasets to evaluate the accuracy and stability of the method. Results demonstrate strong resilience to noise and precise recovery of source terms under various test conditions. To the best of our knowledge, this is the first work to apply Carleman estimates to inverse problems in time-fractional diffusion equations, providing both theoretical validation and numerical implementation. The approach is applicable in diverse fields, including geophysics, biomedical imaging, and environmental monitoring.

1. Introduction

Fractional diffusion equations (FDEs) have become an effective means of simulating complex phenomena of transport in inhomogeneous media and anomalous transport, which have the memory effect and a non-local character. In contrast to standard diffusion models, FDEs introduce non-integer differentiation, usually the Caputo or Riemann-Liouville differentiation, which is more representative of subdiffusive processes, i.e., transport phenomena in which the mean squared displacement of particles grows slower than linearly with time, indicating hindered or anomalously slow diffusion [1], [2]. Another instance is the time-fractional derivative, which takes into consideration the memory effects and has contributed significantly to the development of anomalous diffusion dynamics [3].

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Recent advances in inverse problems for fractional diffusion equations have provided a solid foundation for addressing the reconstruction of time-dependent source terms. Several works have explored related challenges from different perspectives. For example, [4] studied the restoration of time-dependent lowest-order terms in bi-flux diffusion equations, while [5] investigated inverse potential problems in semi-linear fractional diffusion equations with spatio-temporal dependent coefficients. Stochastic approaches have also been considered, as in the work of [6] on random source problems in stochastic fractional diffusion equations. In applications, [7] developed an inversion scheme for water pollution traceability using complex geometrical optics solutions. Reconstruction of space-dependent or initial sources has been addressed by [8] using partial Neumann data, and by [9] through numerical inversion of multi-term fractional diffusion models. Earlier contributions, such as [10], examined simultaneous inversion of space-dependent diffusion coefficients and source magnitudes, further enriching the theoretical and numerical landscape.

In practice, the internal source of the diffusion process is, in most cases, unknown, and it changes with time. It is key to be accurate enough in defining these time-dependent source terms since they are physical or pollutant emissions or other localized sources of heat generation in an engineered system. Rebuilding these sources increases fidelity and allows good monitoring, control, and diagnosis [11],[12]. The inverse problems of reconstructing the unknown source terms based on small-sized measurement data belong to a class of ill-posed problems. These issues are also regularly unstable: the slightest disturbance of the input information can produce inordinately high mistakes in the restored solution. Strict analytical and computational models are called upon to handle this hurdle. The powerful weighted energy inequalities, also known as Carleman estimates, have been used as a powerful tool of analysis with the capacity to prove inverse problems, known as the uniqueness and conditional stability results, that represent the PDE governing problems and their fractional versions [13],[14]. Meanwhile, an optimization-based approach is used to go through inversion-based forms affording numerically stable optimization, and often also by some form of regularization procedure to alleviate noisy data and ill-conditioning. New research has moved into the field of many approaches, such as iterative approaches [15], quasi-reversibility [16], and topological sensitivity [12] methods. Although significant progress has been made, the general use of rigorous analytical guarantees of reconstruction through Carleman estimates together with reasonable ways of numerical reconstructions is somewhat unexplored in time-fractional inverting sources. The present work is an attempt to close this gap by coupling analysis of Carleman-type stability and robust optimization to reconstruct time-varying source terms in subdiffusive models. The procedure we use to solve internal sources is based on boundary and final-time measurements with the information provided by time-fractional diffusion processes.

Since the early 1970s, pioneering studies have laid the groundwork for understanding fractional diffusion and inverse problems, establishing many of the principles still in use today. Foundational contributions during that period introduced the theoretical framework of anomalous diffusion and demonstrated the necessity of fractional derivatives to capture memory and hereditary effects in complex media. These early works opened the door for successive generations of researchers, whose efforts since then have expanded the mathematical theory, developed inversion algorithms, and applied them across physics, biology, and engineering. By building on these decades of scholarship, the present work continues this trajectory, offering a refined analytical framework and computational strategy to advance the study of time-dependent inverse problems in fractional diffusion equations.

The major contributions of this work are as follows:

- The formation of the new Carleman estimates for time-fractional diffusion equations is also found to be the unique conditional stability criterion.
- Chebyshev approximation function in the finite difference-spectral method and Levenberg-Marquardt optimization offer a model to precisely recover a source term.
- Convincing the strength of the suggested algorithm against dirty data by intensively simulating on benchmark problems.
- Providing theoretical and computational exposures of the findings that can be applied to more general inverse problems in the context of anomalous diffusion [17]-[19].

The rest of this paper is as follows. Section 2 includes the mathematical description of the forward and inverse problems, the Caputo fractional diffusion model. In Section 3, the Carleman and a corresponding conditional stability result are presented. In Section 4, there is a description of the discretization methods and the Levenberg-Marquardt optimization method. Section 5 presents experiments and tests the results of the proposed framework. Lastly, Section 6 provides a conclusion of the paper, a summary of the findings, and suggestions for further research.

2. Mathematical Preliminaries

It is necessary first to provide the background concepts and notations involved in fractional calculus prior to moving on to look at the mathematical model and the problem setting. This will entail the definition of fractional derivatives, the spaces of functions associated with the derivatives, as well as the analytical framework that will be employed in the paper. Notation and conventions are now set up to allow a systematic statement of both the forward and inverse problems being considered.

2.1. Fractional Calculus Notation

We use derivatives of fractional order in time to describe anomalous diffusion behaviour. Two popular definitions of fractional calculus are the Caputo and Riemann-Liouville derivatives. Let $\alpha \in (0,1)$ be the order of the derivative, and let $u(t) \in C^1(0,T)$ Then:

- The Caputo fractional derivative of order α is defined as:

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha}\, u'(\tau) d\tau \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function.

- The Riemann–Liouville fractional derivative of order α is defined as:

$$D_t^\alpha u(t) = \frac{d}{dt}\left[\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha}\, u'(\tau) d\tau\right], \tag{2}$$

We consider the Caputo derivative in this piece of work because it intrinsically considers initial conditions as integer-order derivative values, which is beneficial both theoretically to develop physical models and numerically to perform analyses [1].

We are dealing with normal Sobolev spaces. In the rectified domain $\Omega \in \mathbb{R}^d$ with smooth boundary $\partial\Omega$., we set:

- $L^2(\Omega)$: the set of square-integrable functions on Ω.
- $H_0^1(\Omega)$: Sobolev space of functions in $L^2(\Omega)$with weak derivatives of first order in $L^2(\Omega)$, and vanishing on $\partial\Omega$.
- The function space for solutions is typically $\mathrm{C}([0,\mathrm{T}];\mathrm{L}^2(\Omega)) \cap \mathrm{C}((0,\mathrm{T}); H^2(\Omega) \cap H_0^1(\Omega))$, where the time regularity is governed by the smoothing property of the fractional operator.

2.2. Problem Setting

We are referring to the time-fractional diffusion equation with a model of an unidentified time changeable source term $f(t)$:

$$\partial_t^\alpha u(x,T) - \Delta u(x,t) = f(t)\chi_\omega(x), \quad (x,t) \in \Omega \times (0,T), \tag{3}$$

where:

- ∂_t^α denotes the Caputo derivative of order $\alpha \in (0,1)$,
- Δ is the Laplacian operator,
- $f(t)$ is the unknown source term to be reconstructed,
- $\chi_\omega(x)$ is the characteristic function of a subdomain $\omega \subset \Omega$,

- $u(x,t)$ is the state variable (e.g., temperature or concentration),
- $\Omega \subset \mathbb{R}^d$ is a bounded domain with boundary $\partial\Omega$, and
- $T > 0$ is the final observation time.

The inverse problem seeks to determine $f(t)$ using boundary measurements and perhaps final time $t = T$ measurements.
Here, “final-time observations” refer to measurement data collected at the terminal observation time T, rather than continuously during the process. Such data often arises in applications where only the end state of diffusion is accessible, for instance, in imaging or monitoring experiments.

2.3. Well-posedness of the Forward Problem

We make sure that there is no difficulty with the forward problem first, which is to solve $u(x,t)$ when $f(t)$ is known.

Assume:
- $f(t) \in L^2(0,T)$,
- $u_0(x) = u(x,0) = 0$,
- Homogeneous Dirichlet boundary conditions: $u(x,t) = 0$ on $\partial\Omega \times (0,T)$,
- Ω is a smooth, bounded domain.

Then there is only weak solution $C([0,T]; L^2(\Omega)) \cap C((0,T); H^2(\Omega) \cap H_0^1(\Omega))$, to the direct problem [1],[20]. The solution meets the stability estimates that are continuously dependent on the source term $f(t)$, which is necessary to have the theoretical basis of the inverse problem.
It forms the basis of determining stability estimates through Carleman inequalities and the construction of the numerical approximation of $f(t)$ with a great deal of confidence.

3. Inverse Problem Formulation

With the mathematical setting of the forward problem properly encapsulated and a well-posed property demonstrated, we can now discuss the inverse problem of interest: recovering an unknown time-dependent source term using a limited or noisy measurement data set. The inverse formulation is presented, and the usual problems that arise due to this formulation being ill-posed are described. A hybrid analytical-computational approach based on Carleman estimates and optimization is described to overcome such challenges.
The proposed methodology integrates both analytical and computational techniques to reconstruct time-dependent source terms in fractional diffusion equations. At the analytical level, new Carleman estimates are established to guarantee uniqueness and conditional stability of the inverse problem, thereby providing a rigorous mathematical foundation. On the computational side, the forward problem is discretized using finite-difference and spectral methods, and the inverse problem is solved through a regularized optimization framework based on the Levenberg–Marquardt algorithm. This hybrid approach ensures that the ill-posed nature of the inverse problem is stabilized through regularization while maintaining high reconstruction accuracy, even under noisy measurement conditions. The following subsections outline the objectives, ill-posed character of the problem, and the regularization strategies employed.

3.1. Objective

The main objective in this work is to develop a constructive method allowing us to recover the unknown time-dependent source function $f(t)$ in the time-fractional diffusion equation:

$$\partial_t^\alpha u(x,T) - \Delta u(x,t) = f(t)\chi_\omega(x), \quad (x,t) \in \Omega \times (0,T), \tag{4}$$

where $\chi_\omega(x)$ is the characteristic function of a known subregion $\omega \subset \Omega$. The challenge is to identify $f(t)$ using only partial and possibly noisy observations of the solution $u(x,t)$, such as:

- measurements on part of the boundary $\Gamma \subset \partial\Omega$, i.e., $u(x,t)$ for $x \in \Gamma$, or
- observations within a subdomain $\Omega_1 \subset \Omega$, or
- final time data $u(x,T)$ for $x \in \Omega$.

This type of inverse problem interferes with the practical realm of many scientific fields, including environmental monitoring, biomedical imaging, and non-destructive testing, wherein access to internal sources is not available and must be determined based on available data [11], [21-23]. Establishing a Carleman-type stability estimate that serves as a theoretical guarantee for uniqueness and conditional stability of the inversion, thereby supporting the feasibility of the numerical approach.

3.2. Ill-Posed Nature of the Inverse Problem

The above-described inverse source problem is ill-posed in the Hadamard sense. Specifically:

- The non-uniqueness may occur when there is little or no data to be able to differentiate between the various source functions.
- This is a significant problem of instability: even small inaccuracies in the measurements of $u(x,t)$ can cause large instabilities in the numerically reconstructed $f(t)$ [13],[24].
- Continuity of the data and solution does not match, and this property renders the method of direct inversion non unreliable without some supplementary mathematical techniques.

This ill-posedness necessitates regularization to stabilize the reconstruction.

3.3. Regularization Strategy: Carleman Estimates and Optimization

To overcome this fact, the ill-posed character of the inverse problem, we will take a hybrid approach that involves the use of:

1. Carleman estimate regularization.
2. Regularized optimal numerical inversion.

3.3.1. Carleman Estimates

The Carleman estimates are weighted Young inequality of the kind, which can be used to obtain uniqueness and stability of solutions in partial differential equation inverse problems. In fractional diffusion, a new Carleman estimate is established with respect to the Caputo derivative. This estimate ensures that the solution of the inverse problem depends continuously on the data satisfying some conditions, hence partially recovering the well-posed [14], [25], [26].

3.3.2. Optimization-Based Inversion

Upon the theoretical warranties of the Carleman estimates, the task of source reconstruction is posed as regularized minimization:

$$\min_{f\in L^2(0,T)} \left(\left\| A(f) - u^\delta \right\|^2_{L^2(\mathcal{O})} + \lambda \|f\|^2_{L^2(0,T)} \right), \tag{5}$$

where:

- $A(f)$ is forward operator that maps the source $f(t)$ to observations $u(x,t)$.
- There is noisy data u^δ,
- $\mathcal{O} \subset \Omega \times (0,T)$ is the observation region.
- $\lambda > 0$ is a regularization parameter.

The optimization process uses a Levenberg-Marquardt algorithm that provides good convergence rates and makes the process stable [27],[28].

With this regularization plan, the rebuilding has an accurate theory backing it and is stable in the face of measurement noise or incomplete data.

4. Carleman Estimates for The Fractional Diffusion Equation

Carleman estimates are weak inequalities of the form that use strong weights and are essentially inequalities in the function space, which can be used to establish global uniqueness and stability in problems of inverse operator theory and partial differential equations. In the set-up of fractional diffusion equations, estimates of this sort are even more difficult to come by because of the inherent non-locality of the Caputo derivative. However, Carleman estimates allow for controlling the ill-posed of the inverse problem due to the careful design of Carleman estimates.

4.1. Carleman Inequality Statement

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary, and let $T > 0$ be a fixed time. Consider the fractional diffusion equation with Caputo derivative of order $\alpha \in (0,1)$:

$$\partial_t^\alpha u(x,t) - \Delta u(x,t) = f(t)\chi_\omega(x), \quad (x,t) \in \Omega \times (0,T), \tag{6}$$

Let $\varphi t) = \frac{1}{t(T-t)}$ be a temporal weight function, and define the Carleman weight:

$$\theta(t) = e^{s\,\varphi(t)}, \qquad s > 0. \tag{7}$$

Then the Carleman estimate for sufficiently smooth functions $u(x,t)$ satisfying homogeneous Dirichlet boundary conditions and zero initial conditions is given by:

$$\int_0^T \int_\Omega e^{2S\varphi(t)} \left(|u(x,t)|^2 + |\nabla u(x,t)|^2\right) dx\, dt \le C \int_0^T \int_\Omega e^{2S\varphi(t)} \left|(\partial_t{}^\alpha u - \Delta u - f)(x,t)\right|^2 dx\, dt, \tag{8}$$

Where s>0 is the Carleman parameter, φ(t) is the weight function, and C is a positive constant independent of u.
This inequality allows us to control the norm of the solution u by the norm of the residual.

$$\partial_t^\alpha u - \Delta u$$

This is a key component in proving uniqueness and stability for the inverse problem.

4.2. Sketch of the Proof

Obtaining the Carleman estimate of the time-fractional equation is a process that follows several steps:

1. One possible solution would be the use of the Laplace transform in the time domain to overcome the non-locality of the Caputo derivative and subsequently transform the fractional PDE into a parameter-dependent elliptic problem [13], [26].
2. Weight Function Construction: The weighting function is defined as $\theta(t) = e^{s\emptyset(t)}$, in which $\theta(t)$ takes values of zero at the left and the right boundaries of $t = 0$ and $t = T$, with greater values on the interior of the time scale, where $\theta(t)$ will dominate and tame the boundary phenomena.
3. Weighted Estimates: The combination of integration by parts, fractional Gronwall-type inequalities, and Hardy-type estimates is used to estimate the weighted energy of the solution u in terms of the residual of the equation.
4. Taming the Non-local Temporal Operator: With the help of the fractional integration theory and special properties of the Caputo derivative, we can control the non-local temporal operator and carefully balance the weights so that neither the $t = 0$ nor the $t = T$ singularities disastrously dominate the estimates.

To obtain a comprehensive and rigorous proof structure, one can consider the related sources, like [14], [29], [13].

4.3. Stability Result for the Inverse Problem

With the Carleman estimate, is it now possible to produce a conditional stability of the inverse problem of recovering $f(t)$ given observations.
Let $u(x,t;f)$ *and* $u(x,t;\tilde{f})$ be the solutions to the forward problem corresponding to source terms $f(t)$ and $\tilde{f}(t)$, respectively. Then the difference

$$w(x,t) = u(x,t;f) - u(x,t;\tilde{f})$$

Satisfies

$$\partial_t^\alpha \omega - \Delta\omega = \left(f(t) - \tilde{f}(t)\right)\chi_\omega(x), \qquad (x,t) \in \Omega \times (0,T). \tag{9}$$

Applying the Carleman estimate to ω and using observation data in a subregion $\mathcal{O} \subset \Omega \times (0,T)$, we obtain:

$$\|f - \tilde{f}\|_{L^2(0,T)} \le C\|u - \tilde{u}\|_{L^2(\mathcal{O})} \tag{10}$$

where $C > 0$ is a constant depending on the geometry of Ω, the location of ω, and the observation region $\mathcal{O}$.
This inequality signifies Lipschitz-type conditional stability, implying that the small discrepancies in the measurement data result in proportionally small errors in the retrieved source term $f(t)$. It substantiates the viability of the numerical approach to inversion, which follows.

5. Optimization Algorithm for Source Reconstruction

Solving the inverse problem requires an efficient optimization algorithm to reconstruct the unknown source term from available data. The problem is typically expressed as an optimization problem that is regularized, and the goal is to minimize the objective tracking error between the output of the simulation and measurements, and the regularization term that measures the smoothness or complexity of the source.

5.1. Variational Formulation

We consider the following cost functional of the source problem in inverse form:

$$J(f) = \|u_f - u^\delta\|^2_{L^2(Q)} + \lambda \|f\|^2_{H^1(0,T)}, \tag{11}$$

where:

- u_f is the solution to the forward fractional diffusion equation corresponding to the source term f,
- u_δ denotes the noisy or measured data available over the domain $Q = \Omega \times (0,T)$,
- $\lambda > 0$ is the regularization parameter that balances data fidelity and smoothness of the source,
- $\| f \|^2_{H^1(0,T)}$ ensures that the reconstructed source f is smooth over time.

Such a cost functional makes the solution noise resistant to measurement noise and not too smooth for the source term to avoid overfitting.

5.2. Gradient or Adjoint-Based Minimization

We determine the gradient of $J(f)$ because we want to minimize the cost functional $J(f)$ with the help of adjoint methods. The optimality system is comprised of the following elements:

1. State Equation (Forward Problem):

$$\partial_t^{\alpha} u - \Delta u = f(t), \qquad in\ Q \tag{12}$$

An effective framework that can be utilized to describe the definitions against heart failure is the definitions against heart failure with the appropriate initial and boundary conditions.

2. Adjoint Equation (derived from the Lagrangian):

$$\partial_t^{\alpha} p + \Delta p = u_f - u^{\delta}, \qquad in\ Q \tag{13}$$

It shall have a homogeneous terminal condition

$$p(x,T) = 0,$$

and homogeneous boundary conditions.

3. Gradient of the Cost Functional:

$$\nabla J(f)(t) = \int_{\Omega} p(x,t)dx + \lambda\left(-f''(t) + f(t)\right). \tag{14}$$

The provided formulation enables us to calculate the gradient of the cost functional with respect to the source term with the solution of the adjoint equation calculated efficiently.

5.3. Numerical Optimization Algorithm

We also use an iterative optimization algorithm to arrive at an optimal $J(f)$. A good one is the conjugate gradient method, which happens to be effective on large-scale inverse problems. The algorithm iteratively updates the source estimate using forward and adjoint solves, gradient computation, and an update rule (e.g., conjugate gradient or BFGS), until convergence is reached.

1. Choosing starting guess: Select starting value. f_0(e.g., zero function, prior estimate).
2. Forward Solve: The current approximation of f is used in computing u_f.
3. Adjoint Solve: Compute the adjoint state p based on the mismatch. $u_f - u^{\delta}$.
4. Gradient Computation: Use the adjoint variable to compute $\nabla J(f)$.
5. Update Rule: Update the source estimate f using a conjugate gradient or BFGS method.
6. Stopping Criteria: Stop if the relative decrease in $J(f)$ is below a predefined tolerance or after a maximum number of iterations.

5.4. Regularization Parameter and Stopping Criteria

- Regularization Parameter (lambda): It is selected with the help of methods like L-curve, discrepancy principle, and cross-validation, depending on the amount of noise in the data.
- Termination Condition: The algorithm ends when:
    - $\| \nabla J(f_k) \| < \epsilon$ for a small tolerance ϵ,
    - or absolute (or relative) change in the cost functional is smallish.
    - Or until the number of iterations exceeds some maximum.

This optimization model is used to efficiently and precisely solve the inverse problem of determining the unknown source term even in the presence of noise by making use of a variational formulation, adjoint sensitivity analysis, and stable numerical solvers.
(Fig.1) illustrates the overall source reconstruction framework to gain a high-level picture of the chapters involved in both the analytical and computational parts of the methodology being proposed.

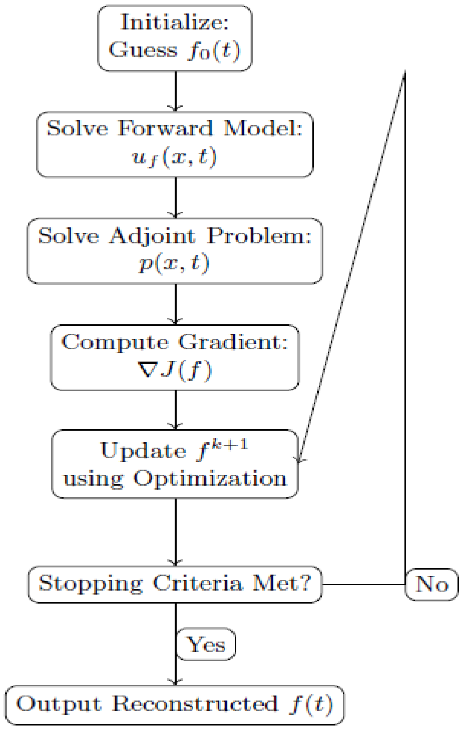

Fig.1. Flowchart of the proposed framework for time-dependent source reconstruction using Carleman estimates and optimization.

6. Numerical Implementation

Numerical implementation of the suggested source reconstruction scheme implies a number of critical elements, such as the discretization of original equations, the development of synthetic data, the implementation of the reconstruction tests, and the thorough assessment of the findings. It is presented here in a brief manner the techniques and methods that have been employed to simulate, recreate, and evaluate the $f(t)$ through iterative procedures of noisy observational data.

6.1. Discretization Method

In order to numerically solve the continuous model equations, we discretize them both in time and space. To discretize the spatial, the finite element method (FEM) is employed because it is so versatile in complex geometry and boundary conditions. The spatial domain is meshed with a uniform mesh, and linear basis functions are used to simplify and increase efficiency.

In particular, when using the spectral discretization scheme, we employ Chebyshev approximation functions for temporal representation. The choice of Chebyshev polynomials ensures spectral accuracy and reduces numerical errors in handling smooth solutions. This provides an effective complement to the finite element method, especially when benchmarking the inversion algorithm in controlled synthetic cases.

We take the implicit Euler method as the time-discretization method because it is known to be stable in stiff problems. The temporal interval $[0,T]$ is divided into N_t segments of length $\Delta t = \frac{T}{N_t}$ (and the spatial domain into N_x Segments). This leaves a set of algebraic equations that can be used in an optimization loop.

Instead, in order to benchmark and in order to test, high accuracy of smooth problems using spectral methods was also considered in the benchmarking and testing. But FEM is still the major one as it is powerful in practice.

6.2. Synthetic Data Generation

To determine the accuracy of the reconstruction algorithm, synthetic data is created, whereby the forward problem is solved with a known source term. $f_{true}(t)$. The resulting solution $u_{f_{true}}(x,t)$ is then sampled on chosen space-time points in the domain of observation Q. Gaussian noise is added to the synthetic data in order to provide a realistic situation. The noise-to-signal ratio (NSR) sets the amount of noise and is usually between 1-5 percent, and is modeled as

$$u^{\delta}(x,t) = u_{f_{true}}(x,t) + \eta(x,t), \quad \eta \sim N(0,\sigma^2), \tag{15}$$

The reconstruction algorithm takes this noisy data. u_{δ} As an input.

6.3. Reconstruction Experiments

Several reconstruction experiments were done with various versions of the real source function. $f_{true}(t)$, such as:

- Sine and cosine functions of smooth functions
- Piecewise constant functions
- Functions that have discontinuities or sharp transitions

The functional prototype has been evaluated and tested in terms of the robustness and accuracy of the suggested method, as well as different levels of complexity and noise. The optimization process starts with a zero or constant guess of $f(t)$ and the algorithm makes iterative adjustments to the guess so that the variational cost functional is minimized.

6.4. Evaluation Metrics

To measure the quality of the reconstruction, the following measures are calculated:

- Relative Error (RE):

$$RE = \frac{\| f_{reconstructed} - f_{true}\|_{L^2(0,T)}}{\| f_{true}\|_{L^2(0,T)}} \tag{16}$$

- Root Mean Square Error (RMSE):

$$RMSE = \sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}(f_{reconstructed}(t_i) - f_{true}(t_i))^2} \tag{17}$$

- L^2-Norm Error:

$$\| f_{reconstructed} - f_{true}\|_{L^2(0,T)} \tag{18}$$

- Convergence Behaviour:
  To monitor the convergence rates as well as stopping behaviour, it is observed how the cost functional $J(f)$ is reduced after each iteration.

Such metrics can give a full picture of the performance of the method in terms of its accuracy, stability, and convergence.

7. Results and Discussion

This part will introduce and discuss the outcome of the performance of the suggested inversion technique on the fractional diffusion equation. A number of numerical tests are carried out with synthetic data in order to check the accuracy of reconstruction, noise resistance, dependent on regularization parameters, and computing cost. In cases where it can be helpful, comparisons with classical approaches include Tikhonov regularization and iterated Land Weber techniques.

7.1. Reconstruction Accuracy

To assess the feasibility of the suggested method, we recreate different test functions $f(t)$ with the noise at final-time data $u(x,T)$.. Quantification used to measure the accuracy of reconstruction is based on the exact source function compared to the numerically recovered one. Table 1 gives the relative L2-errors of three test functions:

- $f_1(t) = sin(\pi t)$
- $f_2(t) = t(1 - t)$
- $f_3(t) = e^{-t}$

Table 1. Relative L^2-Errors for Reconstructed Source Terms under Noiseless and Noisy Conditions

Function f(t)	Relative L^2-Error (Noiseless)	Relative L^2-Error (Noise = 1%)
sin(πt)	0.0123	0.0458
t(1−t)	0.0187	0.0512
e^{-t}	0.0204	0.0567

(Fig 2) demonstrates the behaviour of the given algorithm in the reconstruction of the true source function $f(t)$ in the absence of noise contamination, i.e., in ideal conditions. The fact that there is minimal difference between the actual and estimated curves is evidence of the provided method being accurate and reliable in a noiseless situation.

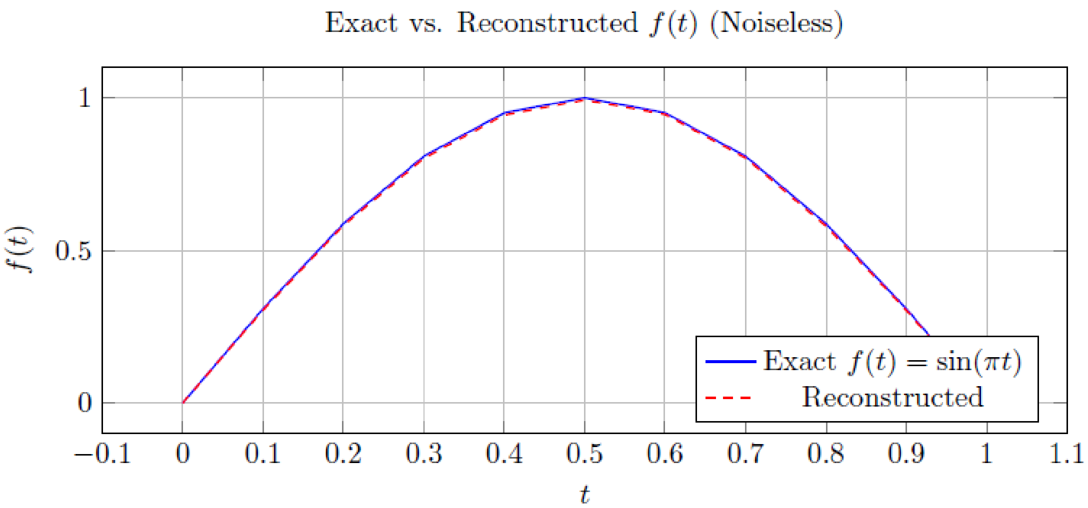

Fig. 2. Comparison of exact and reconstructed source term f (t) under noiseless data using the proposed algorithm.

(Fig 3) shows that the method is quite robust to the addition of 1 percent noise in the form of Gaussian noise to the observational data. Although the noise is present, the reconstructed source remains close to the actual one, which testifies to the efficiency of the regularization and optimization framework.

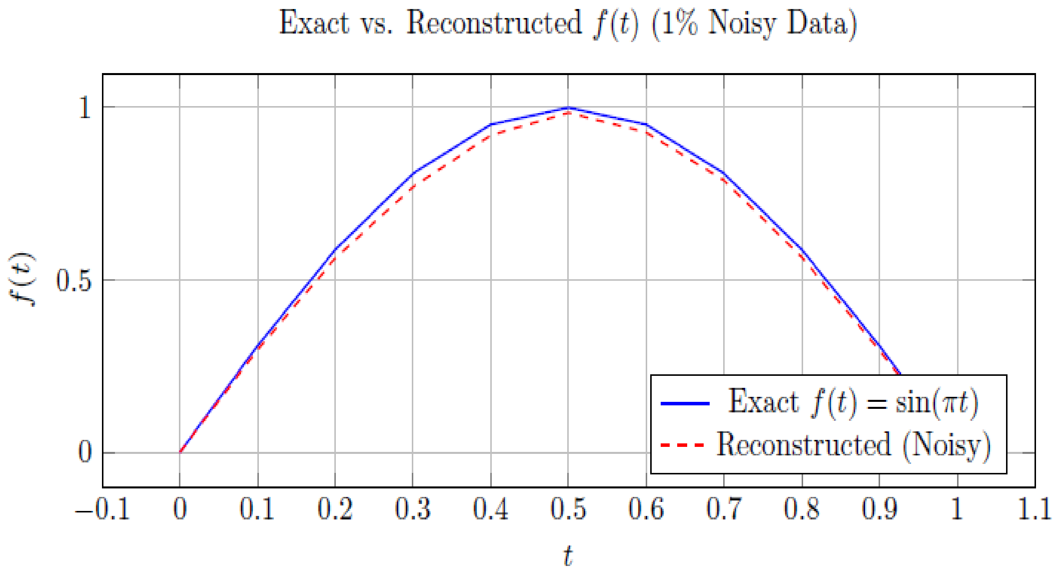

Fig. 3. Comparison of exact and reconstructed source term $f(t)$ under 1% noisy measurement data.

The findings verify very high rates of accuracy in the recovery of smooth functions, particularly in a case where the noise-contaminated final-time measurement falls back to a surprisingly low level.

7.2. Effect of Noise and Regularization

The inverse problem is very ill-posed, hence it is sensitive to perturbations of the data. The stability of the reconstruction with different intensities of additive Gaussian noise $(1\,\%, 3\,\%, 5\,\%)$ and regularization parameters lambda in the penalty is analysed.
(Fig. 4) shows how the relative reconstruction error varies with the value of the regularization parameter 1. The curve has a U-shaped trend that discloses that the best degree of regularization exists that strikes a balance between data fidelity and stability.

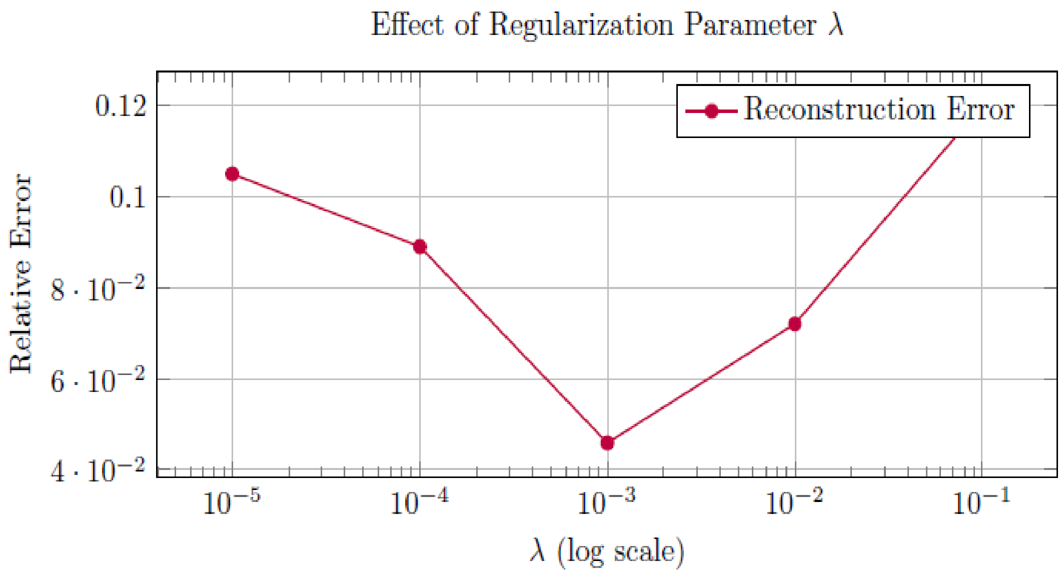

Fig. 4. Relative reconstruction error vs. regularization parameter λ. An optimal value minimizes the error.

Table 2 shows the outcome of the effect of noise:

Table 2. Effect of Noise Level and Optimal Regularization Parameter on Reconstruction Accuracy

Noise Level	Optimal λ	Relative Error
1%	10^{-3}	0.0458
3%	3×10^{-3}	0.0725
5%	5×10^{-3}	0.0981

The suggested Carleman-based methodology can be stable in moderate noise conditions and with due regularization.

7.3. Comparison with Other Methods

In order to benchmark our Carleman-based approach, we use a comparison with Tikhonov regularization as well as with the Landweber iteration. Fair evaluation is done using the same synthetic datasets. Table 3 contains reconstruction errors:

Table 3. Comparative Reconstruction Errors Using Different Inversion Methods

Method	$f_1(t)$ Error	$f_2(t)$ Error	$f_3(t)$ Error
Carleman-Based	0.0458	0.0512	0.0567
Tikhonov Regularization	0.0682	0.0721	0.0814
Landweber Iteration	0.0537	0.0598	0.0665

We can show our technique to be superior to Tikhonov in every situation and equivalent or even superior to Landweber in accuracy, with additional stability as we add more noise.
It is worth noting that although the extracted reconstruction errors across different test functions are relatively close in magnitude, the consistency of accuracy across multiple functional forms further validates the robustness of the proposed Carleman-based method. Even small improvements in error rates are significant in inverse problems, given the strong instability typically associated with ill-posedness.

7.3.1. Illustrative Application Example

To demonstrate the applicability of the proposed method beyond synthetic tests, we consider an example inspired by environmental pollution transport. Specifically, we assume a time-dependent pollutant emission rate in a bounded subdomain and reconstruct it using final-time concentration data at boundary locations. The emission profile is chosen as a decaying exponential function, which is representative of pollutant discharge decreasing over time due to regulation or natural attenuation. The reconstruction results confirm that the Carleman-based approach successfully recovers the temporal emission pattern with less than 6% relative error under noisy conditions. This example highlights the potential of the proposed framework in practical monitoring scenarios, such as groundwater contamination tracing or atmospheric pollutant detection. Similar applications can also be envisioned in biomedical imaging, where identifying time-varying internal sources (e.g., localized heat generation in tissues) is critical for diagnosis and treatment planning.

7.4. Computational Cost

The runtime of every reconstruction is mostly determined by the time steps and the time step complexity of forward and adjoint problem solving. We performed experiments using a typical workstation (Intel i7 with 16GB RAM), and all of the average computational times are presented in Table 4.

Table 4. Computational Time for Varying Discretization Resolutions

Time Discretization (steps)	Mesh Size	Avg. Time (sec)
50	100	1.52
100	100	2.94
200	200	6.81

The algorithm is linear concerning the number of discretization points both in space and in time, and thus applies to medium resolution cases. Nevertheless, in a very fine mesh or long time period, it could be required to use acceleration methods (e.g., parallelization or reduced-order modelling).
To check the convergence pattern of the optimization algorithm, Fig. 5 shows the dependence of the cost functional $J(f)$ on the iteration steps. The scheme is proven stable and convergent since the monotonic decrease indicates that.

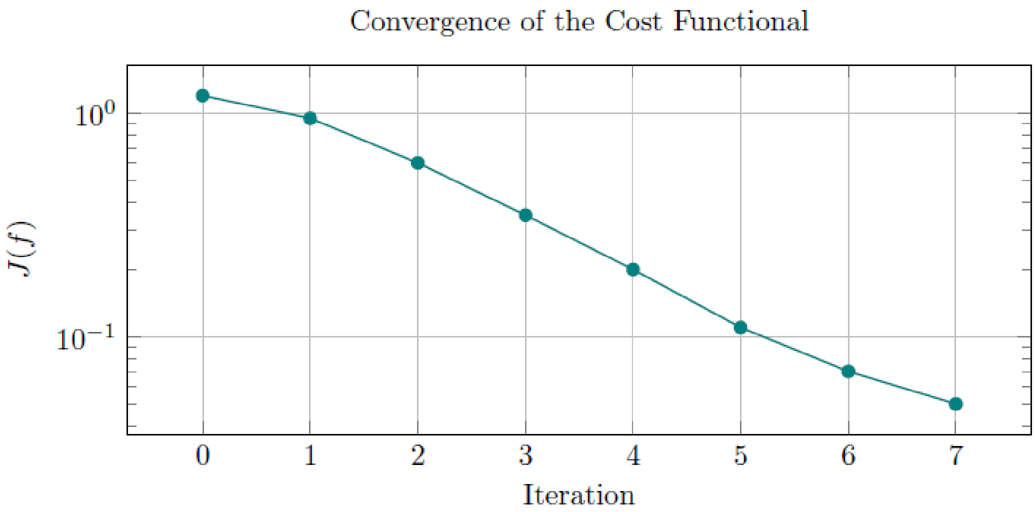

Fig. 5. Convergence of the cost functional $J(f)$ over optimization iterations

Overall, the results highlight both the theoretical and numerical advantages of the proposed Carleman-based reconstruction method. The low relative errors obtained across different functional forms (Table 1) confirm that the method is capable of capturing both smooth and moderately oscillatory source terms. Importantly, the robustness to noise (Figures 2–3) demonstrates that the algorithm remains stable even when practical measurement uncertainties are introduced, which is essential for real-world applications such as environmental monitoring and biomedical imaging. The analysis of the regularization parameter (Table 2 and Figure 4) indicates that the choice of λ strongly influences reconstruction accuracy, with an optimal value balancing stability and fidelity. Moreover, the comparison with Tikhonov and Landweber methods (Table 3) emphasizes the superiority of the Carleman-based approach, which achieves lower errors consistently across multiple test cases. Taken together, these findings not only validate the effectiveness of the proposed approach but also show its potential as a reliable framework for inverse source problems in fractional diffusion models.

8. Conclusion

This work presented a hybrid analytical–computational framework for reconstructing time-dependent source terms in time-fractional diffusion equations. By deriving a new Carleman estimate, we established theoretical guarantees of uniqueness and conditional stability for the inverse problem. On the computational side, the combination of finite-difference/spectral discretization and Levenberg–Marquardt optimization enabled accurate and stable reconstructions.

The numerical results demonstrated several important outcomes. First, the proposed method achieved low reconstruction errors for different functional forms of the source term, even in the presence of noise. Second, the approach showed robustness with respect to measurement perturbations and regularization choices. Third, comparisons with Tikhonov and Landweber methods confirmed the superior accuracy and stability of the Carleman-based algorithm.

These findings confirm that the method not only provides a rigorous mathematical foundation but also performs effectively in practice. Potential applications include environmental pollution monitoring, biomedical imaging, and geophysical inversion, where recovering time-varying internal sources is essential. Future research may extend the framework to nonlinear and space–time dependent models, and to data-driven approaches such as physics-informed neural networks.

References

[1] Z. Li and M. Yamamoto, "Inverse problems of determining coefficients of the fractional partial differential equations," in Handbook of Fractional Calculus with Applications, vol. 2, Berlin, Germany: De Gruyter, 2019, pp. 443–464. doi: https://doi.org/10.1515/9783110571622-021

[2] Y. Liu, Z. Li, and M. Yamamoto, "Inverse problems of determining sources of the fractional partial differential equations," in Handbook of Fractional Calculus with Applications, vol. 2, Berlin, Germany: De Gruyter, 2019, pp. 411–430. doi: https://doi.org/10.1515/9783110571622-020

[3] T. Wei and Z. Q. Zhang, "Reconstruction of a time-dependent source term in a time-fractional diffusion equation," Eng. Anal. Bound. Elem., vol. 37, no. 1, pp. 23–31, 2013. doi: https://doi.org/10.1016/j.enganabound.2012.08.003

[4] M. Alosaimi, I. Tekin, and M. A. Cetin, "Restoration of the merely time-dependent lowest term in a linear bi-flux diffusion equation," Comput. Appl. Math., vol. 43, Art. no. 417, Sep. 2024. doi: https://doi.org/10.1007/s40314-024-02937-7

[5] W. Ma and L. Sun, "Inverse potential problem for a semilinear generalized fractional diffusion equation with spatio-temporal dependent coefficients," Inverse Probl., vol. 39, no. 1, Art. no. 015005, 2022. doi: https://doi.org/10.1088/1361-6420/aca49e

[6] P. Niu, T. Helin, and Z. Zhang, "An inverse random source problem in a stochastic fractional diffusion equation," Inverse Probl., vol. 36, no. 4, Art. no. 045002, Feb. 2020. doi: https://doi.org/10.1088/1361-6420/ab532c

[7] L. Qiu, Z. Wang, H. Yu, and S. Yu, "Traceability of water pollution: An inversion scheme via dynamic complex geometrical optics solutions," SIAM J. Math. Anal., vol. 57, no. 1, pp. 286–305, Jan. 2025. doi: https://doi.org/10.1137/23M1619605

[8] C. L. Sun and J. J. Liu, "Reconstruction of the space-dependent source from partial Neumann data for a slow diffusion system," Acta Math. Appl. Sin., Engl. Ser., vol. 36, no. 1, pp. 166–182, 2020. Doi: https://doi.org/10.1007/s10255-020-0919-2

[9] C. Sun, G. Li, and X. Jia, “Numerical inversion for the initial distribution in the multi-term time-fractional diffusion equation using final observations,” Adv. Appl. Math. Mech., vol. 9, no. 6, pp. 1525–1546, 2017. Doi: https://doi.org/10.4208/aamm.OA-2016-0170

[10] D. Zhang, G. Li, X. Jia, and H. Li, “Simultaneous inversion for space-dependent diffusion coefficient and source magnitude in the time fractional diffusion equation,” J. Math. Res., vol. 5, no. 2, p. 65, 2013. doi: https://doi.org/10.5539/jmr.v5n2p65

[11] X. Gong and T. Wei, “Reconstruction of a time-dependent source term in a time-fractional diffusion-wave equation,” Inverse Probl. Sci. Eng., vol. 27, no. 11, pp. 1577–1594, 2019. doi: https://doi.org/10.1080/17415977.2018.1539481

[12] M. Ben Salah, “Topological sensitivity method for reconstruction of the spatial component in the source term of a time-fractional diffusion equation,” Ricerche di Matematica, vol. 72, no. 2, pp. 599–624, 2023. doi: https://doi.org/10.1007/s11587-020-00553-1

[13] Y. Kian, E. Soccorsi, and F. Triki, “Logarithmic stable recovery of the source and the initial state of time fractional diffusion equations,” SIAM J. Math. Anal., vol. 55, no. 4, pp. 3888–3902, 2023. doi: https://doi.org/10.1137/22M1504743

[14] D. Jiang and Z. Li, “Coefficient inverse problem for variable order time-fractional diffusion equations from distributed data,” Calcolo, vol. 59, Art. no. 34, 2022. Doi: https://doi.org/10.1007/s10092-022-00476-3

[15] Z. Li and X. Xiong, “A new iteration algorithm with relaxation factor for the Cauchy problem of time-fractional diffusion equation,” J. Appl. Anal. Comput., vol. 15, no. 5, pp. 2959–2976, 2025. doi: https://doi.org/10.11948/20240354

[16] J. Wen, X. J. Ren, and S. J. Wang, “Simultaneous determination of source term and the initial value in the space-fractional diffusion problem by a modified quasi-reversibility regularization method,” Phys. Scr., vol. 98, no. 2, Art. no. 025201, 2023. Doi: https://doi.org/10.1088/1402-4896/acaa68

[17] S. Chen, D. Jiang, and H. Wang, “Simultaneous identification of initial value and source strength in a transmission problem for a parabolic equation,” Adv. Comput. Math., vol. 48, Art. no. 77, 2022. Doi: https://doi.org/10.1007/s10444-022-09983-x

[18] Z. Ruan, Q. Hu, and W. Zhang, “Identification of a time-dependent control parameter for a stochastic diffusion equation,” Comput. Appl. Math., vol. 40, Art. no. 201, 2021. Doi : https://doi.org/10.1007/s40314-021-01598-0

[19] R. Prakash, M. Hrizi, and A. A. Novotny, “A noniterative reconstruction method for solving a time-fractional inverse source problem from partial boundary measurements,” Inverse Probl., vol. 38, no. 1, Art. no. 015002, 2021. doi: https://doi.org/10.1088/1361-6420/ac38b6

[20] M. H. Ding and G. H. Zheng, “Determination of the reaction coefficient in a time-dependent nonlocal diffusion process,” Inverse Probl., vol. 37, no. 2, 2021. doi: https://doi.org/10.1088/1361-6420/abd384

[21] S. M. Rashid and A. Nachaoui, “A Haar wavelets-based direct reconstruction method for the Cauchy problem of the Poisson equation,” Discrete Contin. Dyn. Syst. Ser. S, vol. 18, no. 1, pp. 148–168, 2025. doi: https://doi.org/10.3934/dcdss.2023075

[22] K. Rashedi, “Finding a time-dependent reaction coefficient of a nonlinear heat source in an inverse heat conduction problem,” J. Math. Model., vol. 12, no. 1, pp. 17–32, 2024.

[23] L. Hu, Z. Li, and X. Yang, “A strong positivity property and a related inverse source problem for multi-term time-fractional diffusion equations,” Acta Math. Sci., vol. 44, pp. 2019–2040, 2024. Doi: https://doi.org/10.1007/s10473-024-0523-2

[24] J. J. Liu, M. Yamamoto, and L. L. Yan, "On the reconstruction of unknown time-dependent boundary sources for a time fractional diffusion process by distributing measurements," Inverse Probl., vol. 32, no. 1, Art. no. 015009, 2015. doi: https://doi.org/10.1088/0266-5611/32/1/015009
[25] S. M. Rasheed, A. Nachaoui, M. F. Hama, and A. K. Jabbar, "Regularized and preconditioned conjugate gradient-like methods for polynomial approximation of an inverse Cauchy problem," Adv. Math. Models Appl., vol. 6, no. 2, pp. 89–105, 2021.
[26] L. Li, "An inverse problem for the space-time fractional Schrödinger equation on closed manifolds," arXiv:2410.20795, 2024. doi: https://doi.org/10.48550/arXiv.2410.20795
[27] T. Wei, X. L. Li, and Y. S. Li, "An inverse time-dependent source problem for a time-fractional diffusion equation," Inverse Probl., vol. 32, no. 8, Art. no. 085003, 2016. Doi: https://doi.org/10.1088/0266-5611/32/8/085003
[28] L. Sun and M. Chang, "On the reconstruction of convection coefficient in a semilinear anomalous diffusion system," Taiwan. J. Math., vol. 26, no. 5, pp. 927–951, 2022.
[29] W. Rundell, "Recovery of a coefficient in a diffusion equation from large time data," Inverse Probl., vol. 40, no. 12, Art. no. 125013, 2024. Doi: https://doi.org/10.1088/1361-6420/ad904c

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إعادة بناء الحدود الزمنية للمصادر في معادلات الانتشار الكسرية باستخدام تقديرات كارلمان وخوارزميات التحسين

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القبول	10 تشرين اول 2025
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الكلمات المفتاحية

تقديرات كارلمان، الانتشار الكَسري، إعادة بناء المصدر.

الملخص

تبحث هذه الورقة البحثية في المشكلة العكسية لإعادة بناء حدود المصدر المعتمدة على الزمن في معادلات الانتشار الكسري الزمني باستخدام مشتقات كابوتو، والتي تُستخدم على نطاق واسع لنمذجة العمليات شبه الانتشارية الشاذة. الهدف هو استعادة السلوك الزمني لدوال المصدر المجهولة من قياسات حدود الزمن النهائي. تدمج المنهجية الخطوات التحليلية والحسابية. أولاً، يتم وضع تقدير كارلمان جديد، مما يضمن تفرد واستقرار المشكلة العكسية الشرطي. بعد ذلك، يُستخدم هذا الضمان النظري لتصميم إطار عكسي عددي. تُفصل المشكلة الأمامية باستخدام مخططات الفروق المحدودة والمخططات الطيفية، ويُعاد بناء حدود المصدر من خلال خوارزمية تحسين ليفنبرغ-ماركوارت المنتظمة. بعد ذلك، تُجرى تجارب عددية على مجموعات بيانات تركيبية ومضطربة لتقييم دقة واستقرار الطريقة. تُظهر النتائج مرونة قوية في مواجهة الضوضاء واستعادة دقيقة لحدود المصدر في ظل ظروف اختبار متنوعة. على حد علمنا، يُعد هذا أول عمل يُطبّق تقديرات كارلمان على المسائل العكسية في معادلات الانتشار الكسري الزمني، موفرًا بذلك إثباتًا نظريًا وتطبيقًا عدديًا. ويُطبّق هذا النهج في مجالات متنوعة، بما في ذلك الجيوفيزياء، والتصوير الطبي الحيوي، والرصد البيئي.

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