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Spin Characters' Decomposition Matrix, S_{24} modulo, p = 7

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A B S T R A C T

In this work, we compute decomposition matrix for the spin characters S_{24}, connected between irreducible spin characters and irreducible modular spin characters, when the field characteristic equal to 7. The method used in this work is $(r, \bar{r})$-inducing in a way to generate projective character for S_{24} by projective character of S_{23} and used maple program to see all the possible of columns and then choose the possible the right columns of them. The aim of this research is to pave the way for finding general relationships and theorems to study irreducible modular spin characters.

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1. Introduction

Symmetric group S_n has a representation group $\bar{S}_n$ with a central $Z = \{-1,1\}$ such that $\bar{S}_n/Z \cong S_n$. The representations which do not have Z in their kernel are called the spin representations of S_n for more information, see [1]. The spin characters of the spin representations of S_n are labeled by the distinct parts of the partitions of n which are called bar partitions of n and denoted by $\langle\alpha\rangle$. In fact, if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is partition of n and $n - m$ is even, then there is one irreducible spin character denoted by $\langle\alpha\rangle^*$ which is self-associate(double), and if $n - m$ is odd, then there are two associate spin characters denoted by $\langle\alpha\rangle$ and $\langle\alpha\rangle'$ see [2]. The decomposition matrix for the spin characters is constructed from the relationships between the irreducible spin characters and the irreducible modular spin characters of the symmetric group; the number of rows and columns corresponds to the number of projective characters and (p, α)-regular classes, respectively [3]. In this study, we found the decomposition matrix of spin characters for S_{24} modulo $p = 7$. The distribution of the spin characters into p-blocks is accomplished using the $(r, \bar{r})$-inducing (restricting) approach [3] [4]. Numerous people conduct research on this subject, have contributed to this field of study [5, 6, 7, 8, 9, 10]. Before we declare any results, let's define certain notations and terminologies. "p.s." is the principal spin character ("p.i.s." indecomposable), "m.s." is means modular spin character ("i.m.s." irreducible), "d_i" is p.i.s. of S_n, "D_i" is p.i.s. of S_{n-1}, and "$\langle\ \ \rangle^{no}$" is the number of i.m.s.

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2. Preliminaries

For the study, some important conclusions were needed.

Theorem 2.1. [7] Any modular (ordinary) character of group G can be written as a linear combination, with non-negative integer coefficients, of the irreducible modular (ordinary) characters of G .

Theorem 2.2. [1] The degree of the spin character $\langle\alpha\rangle = \langle\alpha_1, \dots, \alpha_m\rangle$ is
$\langle\alpha\rangle(1) = 2^{[(n-m)/2]}\,(n!/\prod_{i=1}^{m}(\alpha_i!))\,\left(\prod_{1\le i<j\le m}(\alpha_i-\alpha_j)/(\alpha_i+\alpha_j)\right)$.

Theorem 2.3. [8] Given that b is the number of p-conjugate characters to the irreducible ordinary character χ of G and that B is an ablock of defect one, then:

a. There exists a positive integer number N such that the irreducible ordinary characters lying in the block B can be partitioned into two disjoint classes:
B_1={$\chi \in B \ |b \deg x \equiv N \ mod \ p^a$}, B_2={$\chi \in B \ |b \deg x \equiv -N \ mod \ p^a$}

b. The block B's decomposition matrix has coefficients that are either 1 or 0.

Theorem 2.4. [9] Let G be a group of order $|G| = m_o p^a$, where $(p, m_o) = 0$. If c is a principal character of sub group H of G, then $\deg c \equiv 0 \ mod \ p^a$.

Theorem 2.5. [2] Let p be odd and n be even, then if $p \nmid n$, then $\langle n\rangle = \varphi\langle n\rangle$ and $\langle n\rangle' = \varphi\langle n\rangle'$ are distinct irreducible modular spin characters.

3. Matrix of Decomposition for the Spin Characters

The decomposition matrix for $\bar{S}_{24}$ of degree (183,123), and it is decomposed in to blocks by the p-core of character it consists of 20 blocks which B_1, B_2, are defect three, B_3, B_4 are defect two, B_5 , B_6, B_7 , B_8 are defect one, and the remaining blocks are defects zero, decomposition matrix is equal to $B_1 \oplus B_2 \oplus \dots \oplus B_{20}$

Lemma 3.1. Decomposition matrix for the block B_2 of type double as shown in the Tables 1.

Table 1. Block B_2

Spin characters	Decomposition matrix
⟨23,1⟩*	
⟨22,2⟩*	
⟨21,2,1⟩	
⟨18,3,2,1⟩*	
⟨17,4,2,1⟩*	
⟨16,8⟩*	
⟨16,7,1⟩	
⟨16,5,2,1⟩*	
⟨16,4,3,1⟩*	
⟨15,9⟩*	
⟨15,7,2⟩	
⟨15,6,2,1⟩*	
⟨15,4,3,2⟩*	
⟨14,9,1⟩	
⟨14,8,2⟩	
⟨14,7,2,1⟩*	
⟨14,4,3,2,1⟩	
⟨13,8,2,1⟩*	
⟨12,9,2,1⟩*	
⟨11,10,2,1⟩	
⟨11,9,3,1⟩*	
⟨11,8,3,2⟩*	
⟨11,7,3,2,1⟩	
⟨10,9,4,1⟩*	

	d_{45}	d_{46}	d_{47}	d_{48}	d_{49}	d_{50}	d_{51}	d_{52}	d_{53}	d_{54}	d_{55}	d_{56}	d_{57}	d_{58}	d_{59}	d_{60}	d_{61}	d_{62}	d_{63}	d_{64}	d_{65}	d_{66}
$\langle 10,8,4,2\rangle^*$																						
$\langle 10,7,4,2,1\rangle$																						
$\langle 9,8,7\rangle$																						
$\langle 9,8,6,1\rangle^*$																						
$\langle 9,8,5,2\rangle^*$																						
$\langle 9,8,4,3\rangle^*$																						
$\langle 9,7,5,2,1\rangle$																						
$\langle 9,7,4,3,1\rangle$																						
$\langle 9,5,4,3,2,1$																						
$\langle 8,7,6,2,1\rangle$																						
$\langle 8,7,4,3,2\rangle$																						
$\langle 8,6,4,3,2,1$																						

Proof: Using (2,6)-inducing of p.i.s. method on D_{23} in S_{24} we have

$$\begin{aligned} D_{23}\uparrow^{(0,1)} S_{24} &= \langle 22,1\rangle + \langle 15,8\rangle + 2\langle 15,7,1\rangle^* + 2\langle 14,8,1\rangle^* + 2\langle 12,8,2,1\rangle + 2\langle 9,8,5,1\rangle \uparrow^{(0,1)} S_{24} \\ &= \langle 23,1\rangle^* + \langle 22,2\rangle^* + \langle 16,8\rangle^* + \langle 15,9\rangle^* + 2\langle 16,7,1\rangle + 2\langle 16,7,1\rangle' + 2\langle 15,7,2\rangle + \\ &\quad 2\langle 15,7,2\rangle' + 2\langle 14,9,1\rangle + 2\langle 14,9,1\rangle' + 2\langle 14,8,2\rangle + 2\langle 14,8,2\rangle' + 2\langle 13,8,2,1\rangle^* + \\ &\quad 2\langle 12,9,2,1\rangle^* + 2\langle 9,8,6,1\rangle^* + 2\langle 9,8,5,2\rangle^* \\ &= k_1 \end{aligned}$$

Similarly, using $(r,\bar{r})$-inducing of p.i.s. $D_2, D_3, D_4, D_5, D_{25}, D_7, D_{29}, D_{10}, D_9, D_{11}, D_{12}, D_{13}, D_{14}, D_{15}$, D_{33}, D_{97}, D_{35}, $D_{18}, D_{37}, D_{20}, D_{21}, D_{22}$ of S_{23} to S_{24} we get on k_2, $k_3, \dots, k_{10}$, c_1, k_{11}, k_{12}, c_2, k_{13}, $k_{14}, \dots, k_{21}$ respectively. Then we get the representation matrix in Table 2.

Table 2. Aproximation matrix for B_2

$\langle 23,1\rangle^*$
$\langle 22,2\rangle^*$
$\langle 21,2,1\rangle$
$\langle 18,3,2,1\rangle^*$
$\langle 17,4,2,1\rangle^*$
$\langle 16,8\rangle^*$
$\langle 16,7,1\rangle$
$\langle 16,5,2,1\rangle^*$
$\langle 16,4,3,1\rangle^*$
$\langle 15,9\rangle^*$
$\langle 15,7,2\rangle$
$\langle 15,6,2,1\rangle^*$
$\langle 15,4,3,2\rangle^*$
$\langle 14,9,1\rangle$
$\langle 14,8,2\rangle$
$\langle 14,7,2,1\rangle^*$
$\langle 14,4,3,2,1$
$\langle 13,8,2,1\rangle^*$
$\langle 12,9,2,1\rangle^*$
$\langle 11,10,2,1$
$\langle 11,9,3,1\rangle^*$
$\langle 11,8,3,2\rangle^*$
$\langle 11,7,3,2,1$
$\langle 10,9,4,1\rangle^*$
$\langle 10,8,4,2\rangle^*$
$\langle 10,7,4,2,1$
$\langle 9,8,7\rangle$
$\langle 9,8,6,1\rangle^*$
$\langle 9,8,5,2\rangle^*$
$\langle 9,8,4,3\rangle^*$
$\langle 9,7,5,2,1\rangle$
$\langle 9,7,4,3,1\rangle$
$\langle 9,5,4,3,2,$

⟨8,7,6,2,1⟩																							
⟨8,7,4,3,2⟩																							
⟨8,6,4,3,2,																							
	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	c_1	k_{11}	k_{12}	c_2	k_{13}	k_{14}	k_{15}	k_{16}	k_{17}	k_{18}	k_{19}	k_{20}	k_{21}

Since $k_{10} + c_2 = k_{14} + c_1$ then $k_{14} \subset c_2$ or, $k_{10} \subset c_1$.
Case1: If $k_{14} \subset c_2$ then $c_2 - k_{14} = d_{57}$ and $d_{57} + k_{10} = c_1$.
Case2: If $k_{10} \subset c_1$ then $c_1 - k_{10} = d_{57}$ and, $d_{57} + k_{14} = c_2$.
Therefore, in both instances, we delete c_1 and replace c_2 with d_{57}.
Case3: $k_4 \not\subset k_3$. To prove this, suppose the opposite.
$(k_3 - k_4) \downarrow_{(1,0)} S_{23} = (\langle 21,2,1\rangle + \langle 21,2,1\rangle' + \langle 16,7,1\rangle + \langle 16,7,1\rangle' + \langle 16,5,2,1\rangle^* + \langle 14,8,2\rangle + \langle 14,8,2\rangle' + \langle 13,8,2,1\rangle^* + \langle 12,9,2,1\rangle^* + \langle 9,8,7\rangle + \langle 9,8,7\rangle' + \langle 9,8,6,1\rangle^*) \downarrow_{(1,0)} S_{23} = 2\langle 20,2,1\rangle^* + \langle 21,2\rangle + \langle 21,2\rangle' + 2\langle 15,7,1\rangle^* + 2\langle 16,6,1\rangle^* + \langle 16,7\rangle + \langle 16,7\rangle' + \langle 15,5,2,1\rangle + \langle 15,5,2,1\rangle' + \langle 16,4,2,1\rangle + \langle 16,4,2,1\rangle' + \langle 16,5,2\rangle^* + 2\langle 13,8,2\rangle^* + 2\langle 14,7,2\rangle^* + 2\langle 14,8,1\rangle^* + \langle 12,8,2,1\rangle + \langle 12,8,2,1\rangle' + \langle 13,7,2,1\rangle + \langle 13,7,2,1\rangle' + \langle 13,8,2\rangle^* + \langle 11,9,2,1\rangle + \langle 11,9,2,1\rangle' + \langle 12,8,2,1\rangle + \langle 12,8,2,1\rangle' + \langle 12,9,2\rangle^* + 2\langle 9,8,6\rangle^* + \langle 9,7,6,1\rangle + \langle 9,7,6,1\rangle' + \langle 9,8,5,1\rangle + \langle 9,8,5,1\rangle' + \langle 9,8,6\rangle^* = D_2 + D_3 - D_4 + D_{27} + D_{28} + D_{33} + D_{34} - D_{35} - D_{36} + D_{95} + D_{96}$
is not p.s. for S_{23} (controduction with **Theorem 2.1**) then $k_4 \not\subset k_3$. The similar approach, we demonstrate that $k_6 \not\subset k_1$, $k_{16} \not\subset k_{14}$, $k_{17} \not\subset k_{13}$ and $k_{17} \not\subset d_{57}$. Since $(k_1 - k_6) \downarrow_{(6,2)} S_{23}$, $(k_{14} - k_{16}) \downarrow_{(1,0)} S_{23}$, $(k_{13} - k_{17}) \downarrow_{(1,0)} S_{23}$, $(d_{57} - k_{17}) \downarrow_{(6,2)} S_{23}$ are not p.s. Consequently, Table 2's approximation matrix equals Table 1. Afterward, delete c_1 and swap out c_2 with d_{57}.

Lemma 3.2. The decomposition matrix for the block B_3 of type double as shown in Table 3.

Table 3. Block B_3

spin characters	Decomposition matrix								
⟨20,4⟩*	1								
⟨18,6⟩*	1	1							
⟨14,6,4⟩		1	1						
⟨13,11⟩*		1		1					
⟨13,7,4⟩	1	1	1	1	1				
⟨13,6,4,1⟩*			1		1	1			
⟨13,5,4,2⟩*						1			
⟨12,6,4,2⟩*					1	1	1		
⟨11,7,6⟩	1			1	1			1	
⟨11,6,5,2⟩*				2	1		1	1	1
⟨11,6,4,3⟩*							1		1
⟨9,6,5,4⟩*				2				2	1
⟨7,6,5,4,2⟩								1	
	d_{67}	d_{68}	d_{69}	d_{70}	d_{71}	d_{72}	d_{73}	d_{74}	d_{75}

Proof:. By used (4,4)-inducing of p.i.s. $D_{41}, D_{43}, D_{45}, D_{47}, D_{49}, D_{51}, D_{53}, D_{55}$, D_{57} of S_{23} to S_{24}. Then we get the approximation matrix in Table 4.

Table 4. Approximate matrix B_3

⟨20,4⟩*	1								
⟨18,6⟩*	1	1							
⟨14,6,4⟩		1	1						
⟨13,11⟩*		1		1					
⟨13,7,4⟩	1	1	1	1	1				
⟨13,6,4,1⟩*			1		1	1			
⟨13,5,4,2⟩*						1			
⟨12,6,4,2⟩*					1	1	1		
⟨11,7,6⟩	1			1	1			1	
⟨11,6,5,2⟩*				2	1		1	1	1
⟨11,6,4,3⟩*							1		1
⟨9,6,5,4⟩*				2				2	1

⟨7,6,5,4,2⟩								1	
	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9

Now on $(7,\alpha)$-regular classes we have:

1. $\langle 14,6,4\rangle = \langle 14,6,4\rangle'$
2. $\langle 13,7,4\rangle = \langle 13,7,4\rangle'$
3. $\langle 11,7,6\rangle = \langle 11,7,6\rangle'$
4. $\langle 7,6,5,4,2\rangle = \langle 7,6,5,4,2\rangle'$
5. $\langle 13,6,4,1\rangle^* = \langle 13,5,4,2\rangle^* + \langle 13,7,4\rangle - \langle 13,11\rangle^* - \langle 20,4\rangle^*$
6. $\langle 11,6,5,2\rangle^* = \langle 11,6,4,3\rangle^* + \langle 11,7,6\rangle + \langle 13,11\rangle^* - \langle 18,6\rangle^*$
7. $\langle 9,6,5,4\rangle^* = \langle 11,6,4,3\rangle^* + 2\langle 7,6,5,4,2\rangle - \langle 12,6,4,2\rangle^* + \langle 13,6,4,1\rangle^* - 2\langle 13,11\rangle - \langle 14,6,4\rangle + 3\langle 8,6\rangle^* - 3\langle 20,4\rangle$
8. $\langle 11,7,6\rangle = \langle 7,6,5,4,2\rangle + \langle 13,7,4\rangle - \langle 14,6,4\rangle$

Since the number of i.m.s is equal to or fewer than the number of spin characters, Table 4 can only have a maximum of 17 columns. However, Table 4 only has a maximum of 9 columns because it has 8 equations that correspond to the spin characters of S_{24} in B_3. And because $k_i - k_j$ is not p.s. to S_{24} $\forall 1 \leq i < j \leq 9$, and $k_1, k_2, \cdots k_9$ are linearly independent. Table 4 therefore equals Table 3.

Lemma 3.3. The decomposition matrix for the block B_4 of type associate as shown in the Tables 5.

Table 5. Block B_4

Spin characters	Decomposition matrix																	
⟨19,4,1⟩																		
⟨19,4,1⟩′		1																
⟨18,5,1⟩			1															
⟨18,5,1⟩′		1																
⟨15,5,4⟩			1		1													
⟨15,5,4⟩′						1												
⟨14,5,4,1⟩*					1		1	1										
⟨12,11,1⟩									1									
⟨12,11,1⟩′				1						1								
⟨12,8,4⟩	1		1						1		1							
⟨12,8,4⟩′		1		1		1						1						
⟨12,7,4,1⟩*					1	1		1				1	1	1				
⟨12,5,4,2,1⟩													1					
⟨12,5,4,2,1⟩′								1						1				
⟨11,8,5⟩											1				1			
⟨11,8,5⟩′		1										1						
⟨11,7,5,1⟩*									1	1	1	1			1	1		
⟨11,5,4,3,1⟩													1				1	
⟨11,5,4,3,1⟩′																		1
⟨8,7,5,4⟩*										1								
⟨8,6,5,4,1⟩																		
⟨8,6,5,4,1⟩′																		
	d_{76}	d_{77}	d_{78}	d_{79}	d_{80}	d_{81}	d_{82}	d_{83}	d_{84}	d_{85}	d_{86}	d_{87}	d_{88}	d_{89}	d_{90}	d_{91}	d_{92}	d_{93}

Proof:. We get an approximate matrix shown in Table 6 through which we find the required matrix by using $(\mathrm{r},\bar{\mathrm{r}})$-inducing of p.i.s. $D_{77}, D_{78}, D_{63}, D_{65}, D_{66}, D_{81}, D_{67}, D_{68}, D_{69}, D_{71}, D_{72}, D_{73}, D_{75}, D_{76}$ of S_{23} to S_{24} we get on $k_1, k_2, k_3, d_{82}, d_{83}, k_4, c_3, c_4, k_5, d_{88}, d_{89}, k_6, d_{92}, d_{93}$ respectively.

Table 6. Approximate matrix B_4

⟨19,4,1⟩	1		
⟨19,4,1⟩′	1		
⟨18,5,1⟩	1	1	
⟨18,5,1⟩′	1	1	
⟨15,5,4⟩		1	1

	k_1	k_2	k_3	d_{82}	d_{83}	k_4	k_5	c_1	c_2	d_{88}	d_{89}	k_6	d_{92}	d_{93}	Y_1	Y_2
$\langle 15,5,4\rangle'$		1	1													
$\langle 14,5,4,1\rangle^*$			2	1	1											
$\langle 12,11,1\rangle$		1				1		1								
$\langle 12,11,1\rangle'$		1				1			1							
$\langle 12,8,4\rangle$	1	1	1			1	1	1	1						a_1	
$\langle 12,8,4\rangle'$	1	1	1			1	1	1	1							a_1
$\langle 12,7,4,1\rangle^*$			2	1	1		2	1	1	1	1				a_2	a_2
$\langle 12,5,4,2,1\rangle$				1						1					a_3	
$\langle 12,5,4,2,1\rangle'$					1						1					a_3
$\langle 11,8,5\rangle$	1						1	1	1			1			a_4	
$\langle 11,8,5\rangle'$	1						1	1	1			1				a_4
$\langle 11,7,5,1\rangle^*$						2	2	3	3	1	1	2	1	1	a_5	a_5
$\langle 11,5,4,3,1\rangle$										1			1		a_6	
$\langle 11,5,4,3,1\rangle'$											1			1		a_6
$\langle 8,7,5,4\rangle^*$						2		2	2			2	1	1	a_7	a_7
$\langle 8,6,5,4,1\rangle$								2				2	1		a_8	
$\langle 8,6,5,4,1\rangle'$									2			2		1		a_8

Since $\langle 19,4,1\rangle \neq \langle 19,4,1\rangle'$ are distinct irreducible modular spin characters then k_1 must split to d_{76}, d_{77}. Since $\langle 15,5,4\rangle \neq \langle 15,5,4\rangle'$ so k_2 or k_3 is split. If k_3 is split to d_{80}, d_{81}. But $\langle 12,11,1\rangle \neq \langle 12,11,1\rangle'$ then k_2 split to, d_{78}, d_{79}. If k_2 is split to, d_{78}, d_{79} and from $(7,\alpha)$-regular classes,

$$\langle 15,5,4\rangle - \langle 18,5,1\rangle + \langle 19,4,1\rangle \neq \langle 15,5,4\rangle' - \langle 18,5,1\rangle' + \langle 19,4,1\rangle' \quad (1)$$

Then k_3 must split to d_{80}, d_{81}. So in both cases we get k_2 and k_3 are splits.

k_5 splits to d_{86}, d_{87} because $\langle 12,8,4\rangle \neq \langle 12,8,4\rangle'$, or there are two columns, Y_1, Y_2. Since $\langle 12,8,4\rangle \downarrow S_{23} = \langle 11,8,4\rangle^{*2} + \langle 12,7,4\rangle^{*10} + \langle 12,8,3\rangle^{*5}$ has 17 of i.m.s and from Table 6 we have $a_1 \in \{0,1,\ldots,10\}$,using the same method we got on $a_3, a_6 \in \{0,1,2,3\}$, $a_5 \in \{0,1,\cdots,12\}$, $a_2 \in \{0,1,\cdots,11\}, a_4, a_7 \in \{0,1,\cdots,10\}$ and $a_8 \in \{0,1,\cdots,7\}$. Take $a_1 \in \{1,\ldots,10\}$ (if $a_1 = 0$ we have contradiction). But $\langle 12,8,4\rangle \downarrow S_{23} \cap \langle 12,5,4,2,1\rangle \downarrow S_{23}$ has no i.m.s so that $a_3 = 0$ also $\langle 12,8,4\rangle \downarrow S_{23} \cap \langle 11,5,4,3,1\rangle \downarrow S_{23}$ has no i.m.s then $a_6 = 0$, so that:

$Y_1 = a_1\langle 12,8,4\rangle + a_2\langle 12,7,4,1\rangle^* + a_4\langle 11,8,5\rangle + a_5\langle 11,7,5,1\rangle^* + a_7\langle 8,7,5,4\rangle^* + a_8\langle 8,6,5,4,1\rangle$,
$Y_2 = a_1\langle 12,8,4\rangle' + a_2\langle 12,7,4,1\rangle^* + a_4\langle 11,8,5\rangle' + a_5\langle 11,7,5,1\rangle^* + a_7\langle 8,7,5,4\rangle^* + a_8\langle 8,6,5,4,1\rangle'$.

To find decomposition matrix we must discuss all probabilities such that the degree $Y_1, Y_2 \equiv 0 \; mod \; 7^3$, that's difficult, so we do the algorithm by maple program in appendix to help us and we find it are equal **30819** probabilities. therefore, we aim to limit the number of possibilities. Since inducing m.s. is m.s. we have:

$$(\langle 12,7,3,1\rangle + \langle 12,10,1\rangle^* - \langle 12,8,3\rangle^* + \langle 19,3,1\rangle^*) \uparrow^{(4,4)} S_{24} \; hence \; a_2 \geq a_1 \quad (2)$$

$$(\langle 12,5,3,2,1\rangle^* - \langle 12,7,3,1\rangle + \langle 12,8,3\rangle^*) \uparrow^{(4,4)} S_{24} \; hence \; a_1 \geq a_2, \therefore a_1 = a_2 \quad (3)$$

$$(\langle 10,8,5\rangle^* - \langle 10,7,5,1\rangle + \langle 10,5,4,3,1\rangle^* + \langle 12,10,1\rangle^*) \uparrow^{(4,4)} S_{24} \; hence \; a_4 \geq a_5 \quad (4)$$

$$(\langle 10,7,5,1\rangle - \langle 10,8,5\rangle^* + \langle 19,3,1\rangle^*) \uparrow^{(4,4)} S_{24} \; hence \; a_5 \geq a_4, \therefore a_4 = a_5 \quad (5)$$

$$(\langle 7,6,5,4,1\rangle^* + \langle 11,5,4,3\rangle + 2\langle 12,11\rangle - \langle 8,6,5,4\rangle) \uparrow^{(0,1)} S_{24} \; hence \; a_8 \geq a_7 \quad (6)$$

$$(\langle 8,6,5,4\rangle - \langle 7,6,5,4,1\rangle^*) \uparrow^{(0,1)} S_{24} \; hence \; a_{12} \geq a_{13}, \therefore a_7 = a_8 \quad (7)$$

$$(\langle 12,6,4,1\rangle + \langle 11,6,5,1\rangle' - \langle 12,7,4\rangle^* + \langle 12,11\rangle + \langle 14,5,4\rangle^* + \langle 19,4\rangle + \langle 19,4\rangle') \uparrow^{(0,1)} S_{24} \; hence \; a_4 \geq a_1 \quad (8)$$

$$(\langle 12,8,3\rangle^* - \langle 10,7,5,1\rangle - \langle 8,7,5,3\rangle + \langle 8,6,5,3,1\rangle^* + \langle 10,5,4,3,1\rangle^* + \langle 12,10,1\rangle^*) \uparrow^{(4,4)} S_{24} \; hence \; a_1 \geq a_4, \therefore a_1 = a_4 \quad (9)$$

$$(\langle 10,8,5\rangle^* - \langle 8,7,5,3\rangle - \langle 12,8,3\rangle^* + \langle 10,5,4,3,1\rangle^* + 2\langle 12,10,1\rangle^* + \langle 15,5,3\rangle^*) \uparrow^{(4,4)} S_{24} \; hence \; 0 \geq a_7, \therefore a_7 = 0 \quad (10)$$

Then we get degree $Y_1, Y_2 \equiv 0 \; mod \; 7^3$ only when $Y_1 + Y_2 = \mathrm{m}(d_{86} + d_{87}), \mathrm{m} \in \{1,2,\ldots,10\}$ which is originally a division of k_5 to d_{86} and d_{87}. In k_6 since $\langle 11,8,5\rangle \neq \langle 11,8,5\rangle'$ then k_6 split to d_{90}, d_{91} or there are two columns, Y_1, Y_2.Take $a_4 \in \{1,\ldots,10\}$ (if $a_4 = 0$ we have contradiction).But $\langle 11,8,5\rangle \downarrow S_{23} \cap \langle 11,5,4,3,1\rangle \downarrow S_{23}$ has no i.m.s so that $a_6 = 0$, so that:

$Y_1 = a_4\langle 11,8,5\rangle + a_5\langle 11,7,5,1\rangle^* + a_7\langle 8,7,5,4\rangle^* + a_8\langle 8,6,5,4,1\rangle$,

$Y_2 = a_4\langle 11,8,5\rangle' + a_5\langle 11,7,5,1\rangle^* + a_7\langle 8,7,5,4\rangle^* + a_8\langle 8,6,5,4,1\rangle'$. and by inducing

$$(\langle 10,8,5\rangle^* + \langle 10,5,4,3,1\rangle^* - \langle 8,7,5,3\rangle + \langle 12,10,1\rangle^*) \uparrow^{(4,4)} S_{24}\ hence\ a_4 \geq a_7 \quad (11)$$

$$(\langle 8,7,5,3\rangle - \langle 10,8,5\rangle^* + \langle 12,8,3\rangle^*) \uparrow^{(4,4)} S_{24}\ hence\ a_7 \geq a_4, \therefore a_4 = a_7 \quad (12)$$

and from Eq.(4), Eq.(5), Eq.(6), Eq.(7) we get degree $Y_1, Y_2 \equiv 0\ mod\ 7^3$ only when $Y_1 + Y_2 =$ m$(d_{90} + d_{91})$ $m \in \{1,2,\ldots 7\}$ which is originally a division of k_6.
Case4: $d_{87} + d_{90} \subset c_1$ suppose it's not, then $\langle 11,7,5,1\rangle^* - \langle 11,8,5\rangle - \langle 11,8,5\rangle' + 2\langle 19,4,1\rangle$ is m.s. for S_{24}, but $(\langle 11,7,5,1\rangle^* - \langle 11,8,5\rangle - \langle 11,8,5\rangle' + 2\langle 19,4,1\rangle) \downarrow_{(0,1)} S_{23}$ is not m.s. for S_{23}
so $d_{87} + d_{90} - c_1 = d_{85}$ (since d_{86}, d_{87}, d_{90}, d_{91} are associate then $d_{86} + d_{91} \subset c_2$ then $d_{86} + d_{91} - c_2 = d_{86}$),
then delete c_1 and c_2 so that we may get $k_4 = d_{85} + d_{86}$. Based on the above, we get Table 5.

Lemma 3.4. The decomposition matrix for the block B_5 is

Table 7. Block B_5

Spin characters	Decomposition matrix		
$\langle 19,5\rangle^*$	1		
$\langle 12,7,5\rangle$		1	
$\langle 12,6,5,1\rangle^*$		1	
$\langle 12,5,4,3\rangle^*$			1
	d_{94}	d_{95}	d_{96}

Proof:. Since
- degree $\{\langle 12,7,5\rangle + \langle 12,7,5\rangle', \langle 12,5,4,3\rangle^*\} \equiv 294$ mod 7^3
- degree $\{\langle 19,5\rangle^*, \langle 12,6,5,1\rangle^*\} \equiv -294$ mod 7^3,

By and by using (3,5)-inducing D_{59}, D_{63}, D_{65} of p.i.s. for S_{23} to S_{24} gives k_1, k_2, k_3 respectively,

and since on $(7, \alpha)$-regular classes:
1. $\langle 12,7,5\rangle = \langle 12,7,5\rangle'$
2. $\langle 12,6,5,1\rangle^* = \langle 12,5,4,3\rangle^* + \langle 12,7,5\rangle - \langle 19,5\rangle^*$

Hence we get the approximation matrix

Table 8. Approximate matrix B_5

Degree				
50778112	$\langle 19,5\rangle^*$	1		
39606927360	$\langle 12,7,5\rangle$	1	1	
77182730240	$\langle 12,6,5,1\rangle^*$		1	1
37626580992	$\langle 12,5,4,3\rangle^*$			1
		k_1	k_2	k_3

Since the number of the i.m.s is equal to or less than the number of the spin characters, the approximation matrix in Table 8 can only have a maximum of 5 columns. It only has three columns at maximum because there are two equations that match to the spin characters of S_{24} in B_5. Then the equivalent Table 8 to Table 7.

Lemma 3.5. The decomposition matrix for the block B_6 is Table 9.

Table 9. Block B_6

Spin characters	Decomposition matrix		
$\langle 18,4,2\rangle$	1		
$\langle 18,4,2\rangle'$		1	
$\langle 11,9,4\rangle$	1		1

$\langle 11,9,4\rangle'$		1		1		
$\langle 11,7,4,2\rangle^*$			1	1	1	1
$\langle 11,6,4,2,1\rangle$					1	
$\langle 11,6,4,2,1\rangle'$						1
	d_{97}	d_{98}	d_{99}	d_{100}	d_{101}	d_{102}

Proof: When we induce D_4, D_8, D_{103}, and D_{104} of p.i.s. for S_{23} through S_{24}, we get on $2k_1$, k_2, d_{101}, and d_{102}, respectively. This results in all i.m.s. being associated in B_6. As a result, Table 10 provides an approximate matrix.

Table 10. Approximate matrix B_6

$\langle 18,4,2\rangle$	1			
$\langle 18,4,2\rangle'$	1			
$\langle 11,9,4\rangle$	1	1		
$\langle 11,9,4\rangle'$	1	1		
$\langle 11,7,4,2\rangle^*$		2	1	1
$\langle 11,6,4,2,1\rangle$			1	
$\langle 11,6,4,2,1\rangle'$				1
	k_1	k_2	d_{101}	d_{102}

Since $\langle 18,4,2\rangle \neq \langle 18,4,2\rangle'$ and $\langle 18,4,2\rangle \downarrow S_{23} \cap \langle 11,7,4,2\rangle \downarrow S_{23}$, $\langle 18,4,2\rangle \downarrow S_{23} \cap \langle 11,6,4,2,1\rangle \downarrow S_{23}$ has no i.m.s so $k_1 = d_{97} + d_{98}$. As $\langle 11,6,4,2,1\rangle \neq \langle 11,6,4,2,1\rangle'$ and B_6 of defect one the $k_2 = d_{99} + d_{100}$, then get Table 9.

Lemma 3.6. The decomposition matrices for the blocks B_7 is

Table 11. Block B_7

Spin characters	Decomposition matrix					
$\langle 16,6,2\rangle$	1					
$\langle 16,6,2\rangle'$		1				
$\langle 13,9,2\rangle$	1		1			
$\langle 13,9,2\rangle'$		1		1		
$\langle 9,7,6,2\rangle^*$			1	1	1	1
$\langle 9,6,4,3,2\rangle$					1	
$\langle 9,6,4,3,2\rangle'$						1
	d_{103}	d_{104}	d_{105}	d_{106}	d_{107}	d_{108}

Proof: We obtain an approximation of a matrix in B_7, which is associated with (2,6)-inducing D_5, D_{16}, and, D_{20} of p.i.s. for S_{23} to S_{24}, in Table 12.

Table 12. Approximate matrix B_7

$\langle 16,6,2\rangle$	[illegible]			a_1	
$\langle 16,6,2\rangle'$	[illegible]				a_1
$\langle 13,9,2\rangle$	[illegible]	[illegible]		a_2	
$\langle 13,9,2\rangle'$	[illegible]	[illegible]			a_2
$\langle 9,7,6,2\rangle^*$		[illegible]	[illegible]	a_3	a_3
$\langle 9,6,4,3,2\rangle$			[illegible]	a_4	
$\langle 9,6,4,3,2\rangle'$			[illegible]		a_4
	k_1	k_2	k_3	Y_1	Y_2

Since $\langle 16,6,2\rangle \neq \langle 16,6,2\rangle'$, so k_3 divided or there are two columns. Suppose there are two columns:
$Y_1 = a_1\langle 16,6,2\rangle + a_2\langle 13,9,2\rangle + a_3\langle 9,7,6,2\rangle^* + a_4\langle 9,6,4,3,2\rangle$,

$Y_2 = a_1\langle 16,6,2\rangle' + a_2\langle 13,9,2\rangle' + a_3\langle 9,7,6,2\rangle^* + a_4\langle 9,6,4,3,2\rangle'$,
to describe columns since B_7 of defect one then we have $a_1, a_2, a_3, a_4 \in \{0,1\}$. Suppose $a_1 = 1$. Since $\langle 16,6,2\rangle \downarrow S_{23} \cap \langle 9,6,4,3,2\rangle \downarrow S_{23}$ has no i.m.s so $a_4 = 0$, As a result of causing m.s. we have:

$$(\langle 12,9,2\rangle - < 16,5,2 >) \uparrow^{(6,2)} S_{24}\ hence\ a_2 \geq a_1 \quad (13)$$

$$(\langle 9,6,4,3,1\rangle - \langle 9,7,6,1\rangle + \langle 13,7,2,1\rangle + < 8,7,6,2 >) \uparrow^{(6,2)} S_{21}\ hence\ a_3 = 0 \quad (14)$$

So $k_3 = d_{103} + d_{104}$. Since $\langle 13,9,2\rangle \neq \langle 13,9,2\rangle'$, $\langle 9,6,4,3,2\rangle \neq \langle 9,6,4,3,2\rangle'$ and B_7 of defect one then k_4, k_5 divided into $d_{105}, d_{106}, d_{107}$ and, d_{108} respectively. Then we get Table 11.

Lemma 3.7. The decomposition matrices for the blocks B_8 in Table 13.

Table 13. Block B_8

Spin characters	Decomposition matrix		
$\langle 15,5,3,1\rangle^*$	1		
$\langle 12,8,3,1\rangle^*$	1	1	
$\langle 10,8,5,1\rangle^*$		1	1
$\langle 8,7,5,3,1\rangle$			1
	d_{109}	d_{110}	d_{111}

Proof:. Since

- degree $\{\langle 8,7,5,3,1\rangle + \langle 8,7,5,3,1\rangle', \langle 10,8,3,1\rangle^*\} \equiv 294 \bmod 7^3$
- degree $\{\langle 15,5,3,1\rangle^*, \langle 10,8,5,1\rangle^*\} \equiv -294 \bmod 7^3$,

By $(3,5)$ −inducing D_{27}, D_{35}, D_{37} of p.i.s. for S_{23} to S_{24} we get on B_8 gives k_1, k_2, k_3 respectively. And since on $(7,\alpha)$-regular classes:

1. $\langle 8,7,5,3,1\rangle = \langle 8,7,5,3,1\rangle'$
2. $\langle 12,8,3,1\rangle^* = \langle 15,5,3,1\rangle^* + \langle 10,8,5,1\rangle^* - \langle 8,7,5,3,1\rangle$

Hence we get the approximation matrix

Table 14. Approximate matrix B_8

degree				
16401330176	$\langle 15,5,3,1\rangle^*$	1		
98407981056	$\langle 10,8,3,1\rangle^*$	1	1	
131210641408	$\langle 10,8,5,1\rangle^*$		1	1
49203990528	$\langle 8,7,5,3,1\rangle$			1
		k_1	k_2	k_3

Due to the fact that the number of i.m.s is equal to or fewer than the number of spin characters, the approximation matrix in Table 14 can only have a maximum of 5 columns. nevertheless only has three columns at most because there are two equations that match to the spin characters of S_{24} in B_8. Additionally, $k_i - k_j$ is not p.s. to S_{24} $\forall 1 \leq i < j \leq 3$. Then, the equivalent Table 14 to Table 13

Theorem 3.8. The decomposition matrix for the block B_1 of type associate in Table 15

Table 15. Block B_1

Spin characters	Decomposition matrix
$\langle 24\rangle$	
$\langle 24\rangle'$	
$\langle 21,3\rangle^*$	
$\langle 20,3,1\rangle$	
$\langle 20,3,1\rangle'$	
$\langle 19,3,2\rangle$	
$\langle 19,3,2\rangle'$	
$\langle 17,7\rangle^*$	
$\langle 17,6,1\rangle$	
$\langle 17,6,1\rangle'$	

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}	d_{20}	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	d_{26}	d_{27}	d_{28}	d_{29}	d_{30}	d_{31}	d_{32}	d_{33}	d_{34}	d_{35}	d_{36}	d_{37}	d_{38}	d_{39}	d_{40}	d_{41}	d_{42}	d_{43}	d_{44}
⟨17,5,2⟩																																												
⟨17,5,2⟩′																																												
⟨17,4,3⟩																																												
⟨17,4,3⟩′																																												
⟨16,5,3⟩																																												
⟨16,5,3⟩′																																												
⟨15,6,3⟩																																												
⟨15,6,3⟩′																																												
⟨14,10⟩*																																												
⟨14,7,3⟩																																												
⟨14,7,3⟩′																																												
⟨14,6,3,1⟩*																																												
⟨14,5,3,2⟩*																																												
⟨13,10,1⟩																																												
⟨13,10,1⟩′																																												
⟨13,8,3⟩																																												
⟨13,8,3⟩′																																												
⟨13,7,3,1⟩*																																												
⟨13,5,3,2,1⟩																																												
⟨13,5,3,2,1⟩′																																												
⟨12,10,2⟩																																												
⟨12,10,2⟩′																																												
⟨12,9,3⟩																																												
⟨12,9,3⟩′																																												
⟨12,7,3,2⟩*																																												
⟨12,6,3,2,1⟩																																												
⟨12,6,3,2,1⟩′																																												
⟨11,10,3⟩																																												
⟨11,10,3⟩′																																												
⟨10,9,5⟩																																												
⟨10,9,5⟩′																																												
⟨10,8,6⟩																																												
⟨10,8,6⟩′																																												
⟨10,7,6,1⟩*																																												
⟨10,7,5,2⟩*																																												
⟨10,7,4,3⟩*																																												
⟨10,6,5,2,1⟩																																												
⟨10,6,5,2,1⟩′																																												
⟨10,6,4,3,1⟩																																												
⟨10,6,4,3,1⟩′																																												
⟨10,5,4,3,2⟩																																												
⟨10,5,4,3,2⟩′																																												
⟨9,7,5,3⟩*																																												
⟨9,6,5,3,1⟩																																												
⟨9,6,5,3,1⟩′																																												
⟨8,7,6,3⟩*																																												
⟨8,6,5,3,2⟩																																												
⟨8,6,5,3,2⟩′																																												
⟨7,6,5,3,2,1⟩*																																												

Proof: Using $(\mathrm{r},\bar{\mathrm{r}})$-inducing of p.i.s. $D_1, D_{41}, D_{42}, D_{77}$, D_4, D_6, D_7, D_5, D_{79}, D_9, $D_{47}, \dots, D_{52}$, D_{13}, D_{82}, D_{83}, D_{16}, $D_{17}, D_{34}, D_{35}, D_{84}, D_{55}, D_{56}, D_{22}, D_{57}, D_{58}$ of S_{23} to S_{24} give us the approximation matrix.

Table 16. Approximate matrix B_1

⟨24⟩
⟨24⟩′

	k_1	d_3	d_4	k_2	k_3	k_4	k_5	k_6	k_7	k_8	d_{19}	d_{20}	d_{21}	d_{22}	d_{23}	d_{24}	k_9	k_{10}	k_{11}	k_{12}	k_{13}	d_{35}	d_{36}	k_{14}	d_{39}	d_{40}	k_{15}	d_{43}	d_{44}	Y_2	Y_1
⟨21,3⟩*																															
⟨20,3,1⟩																															
⟨20,3,1⟩′																															
⟨19,3,2⟩																															
⟨19,3,2⟩′																															
⟨17,7⟩*																															
⟨17,6,1⟩																															
⟨17,6,1⟩′																															
⟨17,5,2⟩																														a_1	
⟨17,5,2⟩′																															a_1
⟨17,4,3⟩																														a_2	
⟨17,4,3⟩′																															a_2
⟨16,5,3⟩																														a_3	
⟨16,5,3⟩′																															a_3
⟨15,6,3⟩																														a_4	
⟨15,6,3⟩′																															a_4
⟨14,10⟩*																														a_5	a_5
⟨14,7,3⟩																														a_6	
⟨14,7,3⟩′																															a_6
⟨14,6,3,1⟩*																														a_7	a_7
⟨14,5,3,2⟩*																														a_8	a_8
⟨13,10,1⟩																														a_9	
⟨13,10,1⟩′																															a_9
⟨13,8,3⟩																														a_{10}	
⟨13,8,3⟩′																															a_{10}
⟨13,7,3,1⟩*																														a_{11}	a_{11}
⟨13,5,3,2,1⟩																														a_{12}	
⟨13,5,3,2,1⟩′																															a_{12}
⟨12,10,2⟩																														a_{13}	
⟨12,10,2⟩′																															a_{13}
⟨12,9,3⟩																														a_{14}	
⟨12,9,3⟩′																															a_{14}
⟨12,7,3,2⟩*																														a_{15}	a_{15}
⟨12,6,3,2,1⟩																														a_{16}	
⟨12,6,3,2,1⟩′																															a_{16}
⟨11,10,3⟩																														a_{17}	
⟨11,10,3⟩′																															a_{17}
⟨10,9,5⟩																														a_{18}	
⟨10,9,5⟩′																															a_{18}
⟨10,8,6⟩																														a_{19}	
⟨10,8,6⟩′																															a_{19}
⟨10,7,6,1⟩*																														a_{20}	a_{20}
⟨10,7,5,2⟩*																														a_{21}	a_{21}
⟨10,7,4,3⟩*																														a_{22}	a_{22}
⟨10,6,5,2,1⟩																														a_{23}	
⟨10,6,5,2,1⟩′																															a_{23}
⟨10,6,4,3,1⟩																														a_{24}	
⟨10,6,4,3,1⟩′																															a_{24}
⟨10,5,4,3,2⟩																														a_{25}	
⟨10,5,4,3,2⟩′																															a_{25}
⟨9,7,5,3⟩*																														a_{26}	a_{26}
⟨9,6,5,3,1⟩																														a_{27}	
⟨9,6,5,3,1⟩′																															a_{27}
⟨8,7,6,3⟩*																														a_{28}	a_{28}
⟨8,6,5,3,2⟩																														a_{29}	
⟨8,6,5,3,2⟩′																															a_{29}
⟨7,6,5,3,2,1⟩*																														a_{30}	a_{30}

All i.m.s. are associated in block B_1, and since $\langle 24\rangle \neq \langle 24\rangle'$ are distinct irreducible modular spin characters , according to (theorem 2.5) $\langle 24\rangle, \langle 24\rangle'$ have the same multiplicity, hence $k_1 = d_1 + d_2$. Since $\langle 19,3,2\rangle \neq \langle 19,3,2\rangle'$ so k_2 or k_3 is split. If k_2 is split to d_5, d_6. But $\langle 17,4,3\rangle \neq \langle 17,4,3\rangle'$ then k_3 split to, d_7, d_8. If k_3 is split to, d_7, d_8 and from $(7, \alpha)$-regular classes,

$$\langle 17,5,2\rangle - \langle 17,4,3\rangle \neq \langle 17,5,2\rangle' - \langle 17,4,3\rangle' \quad (15)$$

Then k_2 is split to d_5, d_6. So in both cases we get k_2 and k_3 are splits.
Since $\langle 17,5,2\rangle \neq \langle 17,5,2\rangle'$ so k_4 or k_5 is split. If k_4 is split to d_9, d_{10}. But $\langle 6,5,3\rangle \neq \langle 6,5,3\rangle'$ then k_5 split to, d_{11}, d_{12}. If k_5 is split to, d_{11}, d_{12} and from $(7,\alpha)$-regular classes,

$$\langle 16,5,3\rangle + \langle 14,7,3\rangle - \langle 15,6,3\rangle \neq \langle 16,5,3\rangle' + \langle 14,7,3\rangle' - \langle 15,6,3\rangle' \quad (16)$$

Then k_4 is split to d_9, d_{10}. So in both cases we get k_4 and k_5 are splits.
Since $\langle 17,4,3\rangle \neq \langle 17,4,3\rangle'$. so k_6 has been divided or has two columns Y_1, Y_2, to explain these columns. Since $\langle 17,5,2\rangle \downarrow S_{23} = \langle 16,5,2\rangle^{*4} + \langle 17,4,2\rangle^{*3} + \langle 17,5,3\rangle^{*2}$ has 9 of i.m.s. and, from Table 16 we have $a_1 \in \{0,1,\dots,5\}$, in the same way we $a_{30} \in \{0,1\}$, $a_5, a_8, a_{25} \in \{0,1,2\}$, $a_2, a_{12}, a_{18}, a_{28} \in \{0,1,2,3\}$, $a_9 \in \{0,1,\dots,4\}$, a_3, $a_{16}, a_{17}, a_{29} \in \{0,1,\dots,5\}, a_4 \in \{0,1,\dots,6\}, a_7, a_{13}, a_{15} \in \{0,1,\dots,7\}$, $a_{20} \in \{0,1,\dots,9\}$, $a_{14}, a_{22}, a_{24} \in \{0,1,\dots,10\}$, $a_{19} \in \{0,1,\dots,11\}$, $a_{11}, a_{23} \in \{0,1,\dots,12\}$, $a_{27} \in \{0,1,\dots 14\}$, $a_{10} \in \{0,1,\dots,15\}$, $a_6, a_{26} \in \{0,1,\dots,16\}$, $a_{21} \in \{0,1,\dots,19\}$. Let $a_2 \in \{1,2,3\}$ (if $a_2 = 0$ contradiction).
Since $\langle 17,4,3\rangle \downarrow S_{23} \cap \langle 14,10\rangle^* \downarrow S_{23}$ has no i.m.s so $a_5 = 0$, the same way we get a_6, a_9, $a_{12}, a_{16}, a_{20}, a_{21}, \dots, a_{30}$ are equal to zero, and since:

$$(\langle 9,7,6,1\rangle + \langle 9,7,6,1\rangle' - \langle 9,8,6\rangle^* + \langle 20,2,1\rangle^*) \uparrow^{(3.5)} S_{24} \; hence \; a_{19} = 0 \quad (17)$$
$$(\langle 19,3,1\rangle^* - \langle 10,8,5\rangle^* + \langle 10,7,5,1\rangle) \uparrow^{(6,2)} S_{24} \; hence \; a_{18} = 0 \quad (18)$$
$$(\langle 12,6,3,2\rangle - \langle 13,5,3,2\rangle) \uparrow^{(1,0)} S_{24} \; hence \; a_{15} \geq a_8 \quad (19)$$
$$(\langle 13,5,3,2\rangle - \langle 12,6,3,2\rangle + \langle 10,6,5,2\rangle) \uparrow^{(1,0)} S_{24} \; hence \; a_8 \geq a_{15}, \therefore a_8 = a_{15} \quad (20)$$
$$(\langle 12,7,3,1\rangle - \langle 14,5,3,1\rangle) \uparrow^{(6,2)} S_{24} hence \; a_{11} \geq a_7 \quad (21)$$
$$(\langle 14,5,3,1\rangle - \langle 12,7,3,1\rangle + \langle 10,7,5,1\rangle) \uparrow^{(6,2)} S_{24} \; hencea_7 \geq a_{11}, \therefore a_7 = a_{11} \quad (22)$$
$$(\langle 14,6,2,1\rangle - \langle 14,4,3,2\rangle) \uparrow^{(5,3)} S_{24} \; hence \; a_7 \geq a_8 \quad (23)$$
$$(\langle 14,4,3,2\rangle - \langle 14,6,2,1\rangle + \langle 14,7,2\rangle^*) \uparrow^{(5,3)} S_{24} \; hence \; a_8 \geq a_7, \therefore a_7 = a_8 \quad (24)$$
$$(\langle 17,5,1\rangle^* - \langle 12,10,1\rangle^* + \langle 8,7,5,3\rangle) \uparrow^{(6,2)} S_{24} \; hence \; a_1 \geq a_{13} \quad (25)$$
$$(\langle 12,10,1\rangle^* - \langle 17,5,1\rangle^* + \langle 19,3,1\rangle^*) \uparrow^{(6,2)} S_{24} \; hence \; a_{13} \geq a_1, \therefore a_1 = a_{13} \quad (26)$$
$$(\langle 16,5,2\rangle^* - \langle 16,4,3\rangle^*) \uparrow^{(5,3)} S_{24} \; hence \; a_1 \geq a_3 \quad (27)$$
$$(\langle 16,4,3\rangle^* - \langle 16,5,2\rangle^* + \langle 16,6,1\rangle^*) \uparrow^{(5,3)} S_{24} \; hence \; a_2 \geq a_1, \therefore a_1 = a_2 \quad (28)$$
$$(\langle 12,9,2\rangle^* - \langle 11,9,3\rangle^* + \langle 10,9,4\rangle^*) \uparrow^{(5,3)} S_{24} \; hencea_1 \geq a_{17} \quad (29)$$
$$(\langle 11,9,3\rangle^* - \langle 12,9,2\rangle^* + \langle 8,7,6,2\rangle + \langle 16,6,1\rangle^*) \uparrow^{(5,3)} S_{24} \; hence \; a_{17} \geq a_1, \therefore a_1 = a_{17} \quad (30)$$

therefore, we only obtain degree $Y_1, Y_2 \equiv 0 \; mod \; 7^3$ when $Y_1 + Y_2 = n_1 k_6 + n_2 k_7, n_1 \in \{1,2,3\}, n_2 \in \{0,1,2\}$, then $k_6 = d_{13} + d_{14}$. In k_7 we have $\langle 16,5,3\rangle \neq \langle 16,5,3\rangle'$. so k_7 divided or there are two columns Y_1, Y_2. Let $a_3 \in \{1,2,,\dots,5\}$ (If $a_3 = 0$ contradiction). Since $\langle 16,5,3\rangle \downarrow S_{23} \cap \langle 14,10\rangle^* \downarrow S_{23}$ has no i.m.s so $a_5 = 0$; similarly,a_{12}, a_{16}, $a_{20}, a_{21}, \dots, a_{30}$ are equal to zero, by Eq.(23), Eq.(24) and since inducing m.s. is m.s., then we have:

$$(\langle 10,6,5,2\rangle + \langle 10,6,5,2\rangle' - \langle 10,7,6\rangle^* + \langle 20,3\rangle) \uparrow^{(1,0)} S_{24} \; hence \; a_{19} = 0 \quad (31)$$
$$(\langle 18,3,2\rangle^* - \langle 10,9,4\rangle^* + \langle 10,7,4,2\rangle) \uparrow^{(5,3)} S_{24} \; hence \; a_{18} = 0 \quad (32)$$
$$(\langle 17,6\rangle - \langle 13,10\rangle + \langle 10,7,6\rangle^*) \uparrow^{(1,0)} S_{24} \; hence \; a_9 = 0 \quad (33)$$
$$(\langle 17,5,1\rangle^* - \langle 12,10,1\rangle^* + \langle 8,7,5,3\rangle) \uparrow^{(6,2)} S_{24} \; hencea_{13} = 0 \quad (34)$$
$$(\langle 12,9,2\rangle^* - \langle 11,9,3\rangle^* + \langle 10,9,4\rangle^*) \uparrow^{(5,3)} S_{24} \; hence \; a_{17} = 0 \quad (35)$$
$$(\langle 12,6,3,2\rangle - \langle 13,5,3,2\rangle) \uparrow^{(1,0)} S_{24} \; hence \; a_{15} \geq a_7 \quad (36)$$
$$(\langle 13,5,3,2\rangle - \langle 12,6,3,2\rangle + \langle 10,6,5,2\rangle) \uparrow^{(1,0)} S_{24} \; hence \; a_7 \geq a_{15}, \therefore a_7 = a_{15} \quad (37)$$
$$(\langle 12,7,3,1\rangle - \langle 14,5,3,1\rangle) \uparrow^{(6,2)} S_{24} \; hence \; a_{11} \geq a_7 \quad (38)$$
$$(\langle 14,5,3,1\rangle - \langle 12,7,3,1\rangle + \langle 10,7,5,1\rangle) \uparrow^{(6,2)} S_{24} \; hence \; a_7 \geq a_{11}, \therefore a_7 = a_{11} \quad (39)$$

$$(\langle 13,7,3\rangle^* - \langle 14,6,3\rangle^*) \uparrow^{(1,0)} S_{24}\ hence\ a_{10} \geq a_4 \quad (40)$$
$$(\langle 14,6,3\rangle^* - \langle 13,7,3\rangle^* + \langle 10,7,6\rangle^*) \uparrow^{(1,0)} S_{24}\ hence\ a_4 \geq a_{10}, \therefore a_4 = a_{10} \quad (41)$$
$$(\langle 15,6,2\rangle^* - \langle 16,5,2\rangle^* + \langle 17,4,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_4 \geq a_3 \quad (42)$$
$$(\langle 16,5,2\rangle^* - \langle 15,6,2\rangle^* + \langle 14,7,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_3 \geq a_4, \therefore a_3 = a_4 \quad (43)$$
$$(\langle 15,5,3\rangle^* - \langle 12,8,3\rangle^* + \langle 12,10,1\rangle^* + \langle 10,8,5\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_3 \geq a_{14}, \quad (44)$$
$$(\langle 12,8,3\rangle^* - \langle 15,5,3\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{14} \geq a_3, \therefore a_3 = a_{14} \quad (45)$$
$$(\langle 11,9,3\rangle^* - \langle 11,7,3,2\rangle + \langle 11,6,3,2,1\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_3 \geq a_7 \quad (46)$$
$$(\langle 11,7,3,2\rangle - \langle 11,9,3\rangle^* + \langle 11,10,2\rangle^* + \langle 18,3,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_7 \geq a_3, \therefore a_3 = a_7 \quad (47)$$

then degree $Y_1, Y_2 \equiv 0\ mod\ 7^3$ only when $Y_1 + Y_2 = m(d_{15} + d_{16}), m \in \{1,2\}$. So $k_7 = d_{15} + d_{16}$. For k_8 we have $\langle 15,6,3\rangle \neq \langle 15,6,3\rangle'$, let $a_4 \in \{1,2,\dots,6\}$. Since $\langle 15,6,3\rangle \downarrow S_{23} \cap \langle 13,5,3,2,1\rangle^* \downarrow S_{23}$ has no i.m.s so $a_{12} = 0$, the same way we get $a_{16}, a_{18}, a_{22}, \dots, a_{25}, a_{27}, a_{28}, a_{29}, a_{30}$ are equal to zero, and since inducing m.s. is m.s. so we have:

$$(\langle 16,4,3\rangle^* - \langle 11,7,3,2\rangle + \langle 11,6,3,2,1\rangle^* + \langle 10,9,4\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{15} = 0 \quad (48)$$
$$(\langle 12,6,3,2\rangle - \langle 13,5,3,2\rangle) \uparrow^{(1,0)} S_{24}\ hence\ a_8 = 0 \quad (49)$$
$$(\langle 9,6,5,2,1\rangle^* - \langle 14,9\rangle + \langle 16,7\rangle) \uparrow^{(5,3)} S_{24}\ hence\ a_5 = 0 \quad (50)$$
$$(\langle 8,7,6,2\rangle - \langle 13,9,1\rangle^* + \langle 14,9\rangle + \langle 16,6,1\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_9 = 0 \quad (51)$$
$$(2\langle 13,10\rangle + 2\langle 7,6,5,3,2\rangle^* + \langle 10,6,4,3\rangle - \langle 9,6,5,3\rangle) \uparrow^{(5,3)} S_{24}\ hence\ a_{26} = 0 \quad (52)$$
$$(\langle 9,6,5,3\rangle - \langle 10,6,5,2\rangle + \langle 10,6,3,2\rangle) \uparrow^{(1,0)} S_{24}\ hence\ a_{21} = 0 \quad (53)$$
$$(\langle 10,6,5,2\rangle + \langle 10,6,5,2\rangle' - \langle 10,7,6\rangle^* + \langle 20,3\rangle + \langle 20,3\rangle') \uparrow^{(1,0)} S_{24}\ hence a_{19}, a_{20} = 0 \quad (54)$$
$$(\langle 17,5,1\rangle^* - \langle 12,10,1\rangle^* + \langle 8,7,5,3\rangle) \uparrow^{(6,2)} S_{24}\ hence\ a_{13} = 0 \quad (55)$$
$$(\langle 12,9,2\rangle^* - \langle 1,9,3\rangle^* + \langle 18,3,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence a_{17} = 0 \quad (56)$$
$$(\langle 16,4,3\rangle^* - \langle 12,9,2\rangle^* + \langle 9,7,5,2\rangle) \uparrow^{(5,3)} S_{24}\ hence\ a_{14} = 0 \quad (57)$$
$$(\langle 15,6,2\rangle^* - \langle 14,6,2,1\rangle + \langle 11,6,3,2,1\rangle^*) \uparrow^{(5,3)} S_{24}\ hence a_4 \geq a_7 \quad (58)$$
$$(\langle 14,6,2,1\rangle - \langle 15,6,2\rangle^* + \langle 16,6,1\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_7 \geq a_4, \therefore a_4 = a_7 \quad (59)$$
$$(\langle 12,7,3,1\rangle - \langle 14,5,3,1\rangle) \uparrow^{(6,2)} S_{24}\ hence\ a_{11} \geq a_4 \quad (60)$$
$$(\langle 14,5,3,1\rangle - \langle 12,7,3,1\rangle + \langle 10,7,5,1\rangle) \uparrow^{(6,2)} S_{24}\ hence\ a_4 \geq a_{11}, \therefore a_4 = a_{11} \quad (61)$$
$$(\langle 19,3,1\rangle^* + \langle 15,5,3\rangle^* - \langle 12,8,3\rangle^* + \langle 10,7,5,1\rangle) \uparrow^{(6,2)} S_{24}\ hence\ a_4 \geq a_{10} \quad (62)$$
$$(\langle 12,8,3\rangle - \langle 15,5,3\rangle) \uparrow^{(6,2)} S_{24}\ hence\ a_{10} \geq a_4, \therefore a_4 = a_{10} \quad (63)$$
$$(\langle 14,7,2\rangle^* - 2\langle 14,6,2,1\rangle + 2\langle 14,4,3,2\rangle) \uparrow^{(5,3)} S_{24}\ hence\ a_6 \geq 2a_4 \quad (64)$$
$$(\langle 13,8,2\rangle^* + \langle 13,7,2,1\rangle + \langle 16,6,1\rangle^* - \langle 14,7,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ 2a_4 \geq a_6, \therefore a_6 = 2a_4 \quad (65)$$

Then degree $Y_1, Y_2 \equiv 0\ mod\ 7^3$ only when $Y_1 + Y_2 = m(d_{17} + d_{18}), m \in \{1,2,\dots,6\}$. So $k_8 = d_{17} + d_{18}$. In k_9 we have $\langle 13,10,1\rangle \neq \langle 13,10,1\rangle'$ so $k_9 = d_{25} + d_{26}$ or there are Y_1, Y_2. Since $\langle 13,10,1\rangle \downarrow S_{23} \cap \langle 13,5,3,2,1\rangle^* \downarrow S_{23}$ has no i.m.s so $a_{12} = 0$, the same way we get a_{15}, $a_{16}, a_{17}, a_{18}, a_{22}, a_{24}, a_{25}, a_{29}, a_{30}$ are equal to zero, and since inducing m.s. is m.s. so we have:

$$(\langle 14,6,3\rangle^* - \langle 13,6,3,1\rangle + \langle 12,6,3,2\rangle) \uparrow^{(1,0)} S_{24}\ hence\ a_{11} = 0 \quad (66)$$
$$(\langle 8,6,4,3,2\rangle^* - \langle 9,8,6\rangle^* + \langle 10,9,4\rangle^* + \langle 14,7,2\rangle^* + \langle 20,2,1\rangle^*) \uparrow^{(5,3)} S_{24}\ hence a_{19} = 0 \quad (67)$$
$$(\langle 10,7,6\rangle^* + \langle 14,6,3\rangle^* - \langle 13,7,3\rangle^*) \uparrow^{(1,0)} S_{24}\ hence\ a_{20} \geq a_{10} \quad (68)$$
$$(\langle 13,7,3\rangle^* - \langle 10,7,6\rangle^* + \langle 7,6,5,3,2\rangle^*) \uparrow^{(1,0)} S_{24}\ hence\ a_{10} \geq a_{20}, \therefore a_{10} = a_{20} \quad (69)$$
$$(\langle 10,7,5,1\rangle - \langle 12,8,3\rangle^* + \langle 15,5,3\rangle^* + \langle 19,3,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{21} \geq a_{14} \quad (70)$$
$$(\langle 10,5,4,3,1\rangle^* + \langle 10,8,5\rangle^* + \langle 12,8,3\rangle^* - \langle 10,7,5,1\rangle) \uparrow^{(6,2)} S_{24}\ hence\ a_{14} \geq a_{21}, \therefore a_{14} = a_{21} \quad (71)$$
$$(\langle 13,9,1\rangle^* - \langle 13,8,2\rangle^* + \langle 13,7,2,1\rangle + \langle 20,2,1\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_9 \geq a_{10} \quad (72)$$

$$(\langle 13,8,2\rangle^* - \langle 13,9,1\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{10} \geq a_9, \therefore a_9 = a_{10} \quad (73)$$
$$(\langle 12,10,1\rangle^* + \langle 12,8,3\rangle^* + \langle 10,8,5\rangle^* + \langle 15,5,3\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{13} \geq a_{14} \quad (74)$$
$$(\langle 12,8,3\rangle^* - \langle 12,10,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{14} \geq a_{13}, \therefore a_{13} = a_{14} \quad (75)$$
$$(\langle 10,5,4,3,1\rangle^* + \langle 8,6,5,3,1\rangle^* + 2\langle 10,8,5\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{27} = 0 \quad (76)$$
$$(\langle 9,6,4,3,1\rangle^* + \langle 7,6,4,3,2,1\rangle - \langle 9,6,5,2,1\rangle^* + \langle 14,7,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{23} = 0 \quad (77)$$
$$(\langle 9,6,5,3\rangle + \langle 12,6,3,2\rangle - \langle 10,6,5,2\rangle) \uparrow^{(1,0)} S_{24}\ hence\ a_{26} \geq a_{13} \quad (78)$$
$$(\langle 10,6,5,2\rangle - \langle 9,6,5,3\rangle + \langle 7,6,5,3,2\rangle^*) \uparrow^{(1,0)} S_{24}\ hence\ a_{13} \geq a_{26}, \therefore a_{13} = a_{26} \quad (79)$$
$$(\langle 10,10,1\rangle^* - \langle 8,7,5,3\rangle + \langle 8,6,5,3,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_9 \geq a_{28} \quad (80)$$
$$(\langle 8,7,5,3\rangle - \langle 12,10,1\rangle^* + \langle 17,5,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{28} \geq a_9, \therefore a_9 = a_{28} \quad (81)$$

then degree$Y_1, Y_2 \equiv 0\ mod\ 7^3$ only when $Y_1 + Y_2 = mk_9, m \in \{1,2,3\}$ then k_9 is split to d_{25}, d_{26}. In k_{11}we have $\langle 13,5,3,2,1\rangle \neq \langle 13,5,3,2,1\rangle'$so $k_{11} = d_{29} + d_{30}$ or there are Y_1, Y_2. Let $a_{12} \in \{1,2,3\}$. (If $a_{12} = 0$ then k_{11} is split). Since $\langle 13,5,3,2,1\rangle \downarrow S_{23} \cap \langle 12,10,2\rangle \downarrow S_{23}$ has no i.m.s so $a_{13} = 0$, the same way we get $a_{17}, a_{18}, a_{19}, a_{28}, a_{29}, a_{30}$ are equal to zero, and since:

$$(\langle 12,4,3,2,1\rangle^* + \langle 13,8,2\rangle^* - \langle 13,7,2,1\rangle) \uparrow^{(5,3)} S_{24}\ hence a_{12} \geq a_{11} \quad (82)$$
$$(\langle 13,7,2,1\rangle - \langle 13,4,3,2,1\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{11} \geq a_{12}, \therefore a_{11} = a_{12} \quad (83)$$
$$(\langle 12,5,3,2,1\rangle^* - \langle 12,7,3,1\rangle + \langle 12,8,3\rangle^*) \uparrow^{(6,2)} S_{24}\ hence a_{16} \geq a_{15} \quad (84)$$
$$(\langle 12,7,3,1\rangle - \langle 12,5,3,2,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{15} \geq a_{16}, \therefore a_{15} = a_{16} \quad (85)$$
$$(\langle 8,6,5,3,1\rangle^* - \langle 8,7,5,3\rangle + \langle 12,10,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{27} \geq a_{26} \quad (86)$$
$$(\langle 8,7,5,3\rangle - \langle 8,6,5,3,1\rangle^* + \langle 10,8,5\rangle^*) \uparrow^{(6,2)} S_{24}\ hence a_{26} \geq a_{27}, \therefore a_{26} = a_{27} \quad (87)$$
$$(\langle 9,7,4,3,2\rangle^* - \langle 13,4,3,2,1\rangle^* + \langle 14,4,3,2\rangle) \uparrow^{(5,3)} S_{24}\ hence\ a_{25} \geq a_{11} \quad (88)$$
$$(\langle 13,4,3,2,1\rangle^* - \langle 9,5,4,3,2\rangle^* + \langle 8,7,6,2\rangle) \uparrow^{(5,3)} S_{24}\ hence\ a_{11} \geq a_{25}, \therefore a_{11} = a_{25} \quad (89)$$
$$(\langle 10,7,6\rangle^* - \langle 13,7,3\rangle^* + \langle 14,6,3\rangle^*) \uparrow^{(1,0)} S_{24}\ hence\ a_{20} \geq a_{11} \quad (90)$$
$$(\langle 13,7,3\rangle^* - \langle 14,6,3\rangle^* + \langle 7,6,5,3,2\rangle^*) \uparrow^{(1,0)} S_{24}\ hence\ a_{11} \geq a_{20}, \therefore a_{11} = a_{20} \quad (91)$$
$$(\langle 10,5,4,3,1\rangle^* - \langle 10,7,5,1\rangle + \langle 10,8,5\rangle^* + \langle 12,10,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{24} \geq a_{21} \quad (92)$$
$$(\langle 10,7,5,1\rangle - \langle 10,7,4,3,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{21} \geq a_{24}, \therefore a_{21} = a_{24} \quad (93)$$
$$(\langle 9,7,4,3\rangle - \langle 9,6,5,2,1\rangle^* + \langle 8,6,4,3,2\rangle^* + \langle 14,7,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{23} \geq a_{22} \quad (94)$$
$$(\langle 9,6,5,2,1\rangle^* - \langle 9,7,4,3\rangle + \langle 11,9,3\rangle^*) \uparrow^{(5,3)} S_{24}\ hence a_{22} \geq a_{23}, \therefore a_{22} = a_{23} \quad (95)$$
$$(\langle 10,6,5,2\rangle + \langle 10,6,5,2\rangle' + \langle 20,3\rangle + \langle 20,3\rangle' - \langle 10,6,4,3\rangle - \langle 10,6,4,3\rangle' - \langle 10,7,6\rangle^*) \uparrow^{(1,0)} S_{24}\ hence\ a_{11} = 0 \quad (96)$$

then $a_{11} = a_{12} = 0$. But that contradiction with since $a_{12} \in \{1,2,3\}$, so k_{11} it's split. In k_{12}we have $\langle 12,10,2\rangle \neq \langle 12,10,2\rangle'$ so $k_{12} = d_{31} + d_{32}$ or there are Y_1, Y_2 . Let $a_{17} \in \{1,2,...,7\}$, since $\langle 12,10,2\rangle \downarrow S_{23} \cap \langle 12,6,3,2,1\rangle \downarrow S_{23}$ has no i.m.s so $a_{16} = 0$, the same way we get $a_{23}, a_{24}, a_{25}, a_{29}$, a_{30} are equal to zero, and since inducing m.s. is m.s. so we have:

$$(\langle 8,6,4,3,2\rangle^* - \langle 8,7,6,2\rangle + \langle 13,9,1\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{28} = 0 \quad (97)$$
$$(\langle 8,6,4,3,2\rangle^* - \langle 9,8,6\rangle^* + \langle 13,8,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{19} = 0 \quad (98)$$
$$(\langle 9,6,4,3,1\rangle^* - \langle 9,7,6,1\rangle + \langle 13,8,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{20} = 0 \quad (99)$$
$$(\langle 10,9,4\rangle^* + \langle 11,6,3,2,1\rangle^* - \langle 11,7,3,2\rangle + \langle 16,5,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{18} \geq a_{15} \quad (100)$$
$$(\langle 11,7,3,2\rangle - \langle 10,9,4\rangle^* + \langle 8,7,6,2\rangle + \langle 18,3,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{15} \geq a_{18} \therefore a_{15} = a_{18} \quad (101)$$
$$(\langle 8,7,5,3\rangle - \langle 12,10,1\rangle^* + \langle 17,5,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{26} \geq a_{13} \quad (102)$$
$$(\langle 12,10,1\rangle^* - \langle 8,7,5,3\rangle + \langle 8,6,5,3,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence a_{13} \geq a_{26}, \therefore a_{13} = a_{26} \quad (103)$$
$$(\langle 10,6,4,3\rangle + \langle 7,6,5,3,2\rangle^* - \langle 10,6,5,2\rangle + 2\langle 13,7,3\rangle^*) \uparrow^{(1,0)} S_{24}\ hence\ a_{22} \geq a_{21} \quad (104)$$
$$(\langle 10,6,5,2\rangle - \langle 10,6,4,3\rangle) \uparrow^{(1,0)} S_{24}\ hence a_{21} \geq a_{22}, \therefore a_{21} = a_{22} \quad (105)$$
$$(\langle 11,10,2\rangle^* - \langle 12,9,2\rangle^* + \langle 13,8,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence a_{17} \geq a_{14} \quad (106)$$

$$(\langle 12,9,2\rangle^* - \langle 11,10,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{14} \geq a_{17}, \therefore a_{14} = a_{17} \quad (107)$$

$$(\langle 11,9,3\rangle^* - \langle 10,7,4,2\rangle + \langle 10,6,4,2,1\rangle^* + \langle 9,8,6\rangle^*) \uparrow^{(5,3)} S_{24}\ hence a_{14} \geq a_{21} \quad (108)$$

$$(\langle 10,7,4,2\rangle - \langle 11,9,3\rangle^* + \langle 16,5,2\rangle^* + \langle 18,3,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence a_{21} \geq a_{14}, \therefore a_{14} = a_{21} \quad (109)$$

then degree$Y_1, Y_2 \equiv 0\ mod\ 7^3$ only when $Y_1 + Y_2 = n_1 k_{12} + n_2 k_{13}, n_1 \in \{1,2\}, n_2 \in \{0,1,2,3\}$, or $n_1 \in \{3,4,5\}, n_2 \in \{0,1,\dots,5-n_1\}$. So k_{12} it's split. For k_{13} $\langle 12,9,3\rangle \neq \langle 12,9,3\rangle'$so $k_{13} = d_{33} + d_{34}$ or there are Y_1, Y_2 . Since $\langle 12,9,3\rangle \downarrow S_{23} \cap \langle 12,6,3,2,1\rangle \downarrow S_{23}$ has no i.m.s so $a_{16} = 0$, the same way we get $a_{23}, a_{24}, a_{25}, a_{27}, a_{29}, a_{30}$ are equal to zero, and since inducing m.s. is m.s. so we have:

$$(2\langle 8,6,5,3,1\rangle^* - \langle 8,7,5,3\rangle + \langle 12,10,1\rangle^*) \uparrow^{(2,6)} S_{24}\ hence\ a_{26}, a_{28} = 0 \quad (110)$$

$$(\langle 13,7,3\rangle^* - \langle 10,7,6\rangle^* + \langle 7,6,5,3,2\rangle^*) \uparrow^{(2,6)} S_{24}\ \ hence\ a_{19}, a_{20} = 0 \quad (111)$$

$$(\langle 11,10,2\rangle^* + \langle 12,9,2\rangle^* - \langle 13,8,3\rangle^*) \uparrow^{(3,5)} S_{24}\ hence\ a_{17} \geq a_{14} \quad (112)$$

$$(\langle 12,9,2\rangle^* - \langle 11,10,2\rangle^*) \uparrow^{(3,5)} S_{24}\ hence\ a_{14} \geq a_{17}, \therefore a_{14} = a_{17} \quad (113)$$

$$(\langle 12,7,3,1\rangle - \langle 12,8,3\rangle^* + \langle 12,10,1\rangle^* + \langle 19,3,1\rangle^*) \uparrow^{(2,6)} S_{24}\ hence\ a_{15} \geq a_{14} \quad (114)$$

$$(\langle 12,8,3\rangle - \langle 12,7,3,1\rangle + \langle 12,5,3,2,1\rangle^*) \uparrow^{(2,6)} S_{24}\ hence\ a_{14} \geq a_{15}, \therefore a_{14} = a_{15} \quad (115)$$

$$(\langle 12,6,3,2\rangle - \langle 10,6,5,2\rangle + \langle 9,6,5,3\rangle) \uparrow^{(0,1)} S_{24}\ hence\ a_{14} \geq a_{21} \quad (116)$$

$$(\langle 10,6,5,2\rangle - \langle 12,6,3,2\rangle + \langle 13,5,3,2\rangle) \uparrow^{(0,1)} S_{24}\ hence\ a_{21} \geq a_{14}, \therefore a_{14} = a_{21} \quad (117)$$

then degree$Y_1, Y_2 \equiv 0\ mod\ 7^3$ only when $Y_1 + Y_2 = mk_{13}, m \in \{1,2,3\}$ then k_{13} is split to d_{33}, d_{34}. In k_{14} we have $\langle 10,9,5\rangle \neq \langle 10,9,5\rangle'$if there are Y_1, Y_2. Let $a_{18} \in \{1,2,3\}$. Since $\langle 10,9,5\rangle \downarrow S_{23} \cap \langle 10,5,4,3,2\rangle \downarrow S_{23}$ has no i.m.s so $a_{25} = 0$, the same way we get a_{30} are equal to zero, and since inducing m.s. is m.s. so we have:

$$(\langle 10,9,4\rangle^* - \langle 9,8,6\rangle^* + 2\langle 7,6,4,3,21\rangle + \langle 13,8,3\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ \ a_{18} \geq a_{19} \quad (118)$$

$$(\langle 9,8,6\rangle^* - \langle 10,9,4\rangle^* + \langle 11,9,3\rangle^*) \uparrow^{(5,3)} S_{24} hence\ a_{19} \geq a_{18}, \therefore a_{18} = a_{19} \quad (119)$$

$$(\langle 8,7,6,2\rangle + \langle 8,7,6,2\rangle' + \langle 7,6,4,3,2,1\rangle + \langle 7,6,4,3,2,1\rangle' - \langle 8,6,4,3,2\rangle^*) \uparrow^{(5,3)} S_{24}$$
$$hence\ 2a_{28} \geq a_{29} \quad (120)$$

$$(\langle 8,6,4,3,2\rangle^* + \langle 9,5,4,3,2\rangle^* - \langle 8,7,6,2\rangle - \langle 8,7,6,2\rangle' + 2\langle 3,9,1\rangle^*) \uparrow^{(5,3)} S_{24}$$
$$hence\ a_{29} \geq 2a_{28}, \therefore a_{28} = 2a_{29} \quad (121)$$

$$(\langle 9,7,6,1\rangle - \langle 9,8,6\rangle^* + \langle 7,6,4,3,2,1\rangle + \langle 13,8,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{20} \geq a_{18} \quad (122)$$

$$(\langle 9,8,6\rangle^* - \langle 9,7,6,1\rangle + \langle 9,5,4,3,2\rangle^* + \langle 13,9,1\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{18} \geq a_{20}, \therefore a_{18} = a_{20} \quad (123)$$

$$(\langle 9,7,4,3\rangle - \langle 10,7,4,2\rangle + \langle 11,7,3,2\rangle) \uparrow^{(5,3)} S_{24}\ hence\ a_{26} \geq a_{21} \quad (124)$$

$$(\langle 10,7,4,2\rangle - \langle 9,7,4,3\rangle + \langle 7,6,4,3,2,1\rangle) \uparrow^{(5,3)} S_{24}\ hence\ a_{21} \geq a_{26}, \therefore a_{21} = a_{26} \quad (125)$$

$$(\langle 8,7,6,2\rangle + \langle 7,6,4,3,2,1\rangle - \langle 9,7,6,1\rangle + \langle 13,7,2,1\rangle) \uparrow^{(5,3)} S_{24}\ hence\ a_{28} \geq a_{18} \quad (126)$$

$$(\langle 9,7,6,1\rangle - \langle 8,7,6,2\rangle) \uparrow^{(5,3)} S_{24}\ hence\ a_{18} \geq a_{28}, \therefore a_{18} = a_{28} \quad (127)$$

$$(\langle 7,9,5,3,2\rangle^* - \langle 10,7,6\rangle^* + \langle 13,7,3\rangle^*) \uparrow^{(1,0)} S_{24}\ \ hence a_{29} \geq 2a_{18} \quad (128)$$

$$(\langle 10,7,6\rangle^* - \langle 7,6,5,3,2\rangle^*) \uparrow^{(1,0)} S_{24}\ \ hence 2a_{18} \geq a_{29}, \therefore a_{29} = 2a_{18} \quad (129)$$

$$(\langle 9,6,4,3,1\rangle^* - \langle 9,6,5,2,1\rangle^* + \langle 13,8,2\rangle^* + \langle 7,6,4,3,2,1\rangle) \uparrow^{(5,3)} S_{24}\ hence\ a_{24} \geq a_{23} \quad (130)$$

$$(\langle 9,6,5,2,1\rangle^* - \langle 9,6,4,3,2,1\rangle^* + \langle 9,5,4,3,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{23} \geq a_{24}, \therefore a_{23} = a_{24} \quad (131)$$

$$(\langle 10,7,5,1\rangle - \langle 10,8,5\rangle^* - \langle 10,5,4,3,1\rangle^* + \langle 19,3,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{21} \geq a_{18} + a_{23} \quad (132)$$

$$(\langle 10,8,5\rangle^* + \langle 10,5,4,3,1\rangle^* - \langle 10,7,5,1\rangle + \langle 12,10,1\rangle^*) \uparrow^{(6,2)} S_{24}$$
$$hence\ a_{18} + a_{23} \geq a_{21},\ \therefore a_{21} = a_{18} + a_{23} \quad (133)$$

$$(\langle 10,7,4,2\rangle - \langle 9,8,6\rangle^* - \langle 10,6,4,2,1\rangle^* + 2\langle 7,6,4,3,2,1\rangle + \langle 13,8,2\rangle^*) \uparrow^{(5,3)} S_{24}$$
$$hence\ a_{22} \geq a_{23} \quad (134)$$

$$(\langle 9,8,6\rangle^* + \langle 10,6,4,2,1\rangle^* - \langle 10,7,4,2\rangle + \langle 11,9,3\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{23} \geq a_{22}, \therefore a_{22} = a_{23} \quad (135)$$

$$(\langle 9,6,5,3\rangle + \langle 9,6,5,3\rangle' - \langle 10,6,5,2\rangle - \langle 10,6,5,2\rangle' - \langle 7,6,5,3,2\rangle^* + \langle 12,6,3,2\rangle$$
$$+ \langle 12,6,3,2\rangle') \uparrow^{(1,0)} S_{24}\ hence\ a_{27} \geq a_{22} + a_{18} \quad (136)$$

$$(\langle 10,6,5,2\rangle + \langle 10,6,5,2\rangle' + \langle 7,6,5,3,2\rangle^* - \langle 9,6,5,3\rangle - \langle 9,6,5,3\rangle') \uparrow^{(1,0)} S_{24}$$
$$hence\ a_{22} + a_{18} \geq a_{27}, \therefore a_{27} = a_{18} + a_{22} \quad (137)$$
$$(\langle 8,6,5,3,1\rangle^* - \langle 8,7,5,3\rangle - \langle 10,8,5\rangle^* + \langle 12,8,3\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{22} \geq a_{21} \quad (138)$$
$$(\langle 8,7,5,3\rangle + \langle 10,8,5\rangle^* - \langle 8,6,5,3,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{21} \geq a_{22}, \therefore a_{21} = a_{22} \quad (139)$$

then from Eq.(133) $a_{21} = a_{18} + a_{23}$ and from Eq.(135),Eq.(139) $a_{21} = a_{22} = a_{23}$, respectively then $a_{23} = a_{18} + a_{23} \Rightarrow a_{18} = 0$. However, given that $a_{18} \neq 0$, so $k_{14} = d_{37} + d_{38}$. For k_{15} we have $\langle 10,5,4,3,2\rangle \neq \langle 10,5,4,3,2\rangle'$, if there are Y_1, Y_2. Let $a_{25} \in \{1,2\}$ and, since:

$$(\langle 8,7,6,2\rangle - \langle 9,5,4,3,2\rangle^* + \langle 11,6,3,2,1\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{28} \geq a_{25} \quad (140)$$
$$(\langle 9,5,4,3,2\rangle^* - \langle 8,7,6,2\rangle + \langle 10,9,4\rangle^* + \langle 12,9,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{25} \geq a_{28}, \therefore a_{25} = a_{28} \quad (141)$$
$$(\langle 10,5,4,3,1\rangle^* - \langle 8,7,5,3\rangle + \langle 10,8,5\rangle^* + \langle 12,8,3\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{24} \geq a_{26} \quad (142)$$
$$(\langle 8,7,5,3\rangle - \langle 10,5,4,3,2,1\rangle^* + \langle 12,5,3,2,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{26} \geq a_{24}\ \therefore a_{24} = a_{26} \quad (143)$$
$$(\langle 8,7,6,2\rangle - \langle 8,6,4,3,2\rangle^* + \langle 9,8,6\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{25} \geq a_{29} \quad (144)$$
$$(\langle 8,6,4,3,2\rangle^* - \langle 8,7,6,2\rangle + \langle 12,9,2\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{29} \geq a_{25}, \therefore a_{25} = a_{29} \quad (145)$$
$$(\langle 8,7,5,3\rangle - \langle 8,6,5,3,1\rangle^* + \langle 10,8,5\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{24} \geq a_{27} \quad (146)$$
$$(\langle 8,6,5,3,1\rangle^* - \langle 8,7,5,3\rangle + \langle 12,8,3\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{27} \geq a_{24}, \therefore a_{24} = a_{27} \quad (147)$$
$$(\langle 8,7,6,2\rangle - \langle 9,7,6,1\rangle + \langle 9,8,6\rangle^* + \langle 11,6,3,2,1\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{25} \geq a_{20} \quad (148)$$
$$(\langle 9,7,6,1\rangle - \langle 8,7,6,2\rangle + \langle 9,8,6\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{20} \geq a_{25}\ \therefore a_{20} = a_{25} \quad (149)$$
$$(\langle 10,5,4,3,1\rangle^* - \langle 10,7,5,1\rangle + \langle 10,8,5\rangle^* + \langle 12,9,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{24} \geq a_{21}, \quad (150)$$
$$(\langle 10,7,5,1\rangle - \langle 10,5,4,3,1\rangle^*) \uparrow^{(6,2)} S_{24}\ hence\ a_{21} \geq a_{24}, \therefore a_{21} = a_{24} \quad (151)$$
$$(\langle 10,7,6\rangle^* - \langle 7,6,5,3,2\rangle^*) \uparrow^{(1,0)} S_{24}\ hence\ a_{30} = 0 \quad (152)$$
$$(\langle 10,6,4,2,1\rangle^* - \langle 9,7,4,3\rangle + \langle 9,8,6\rangle^* + \langle 11,9,3\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{23} \geq a_{22} \quad (153)$$
$$(\langle 9,7,4,3\rangle - \langle 10,6,4,2,1\rangle^* + \langle 11,7,3,2\rangle) \uparrow^{(5,3)} S_{24}\ hence\ a_{22} \geq a_{23}, \therefore a_{22} = a_{23} \quad (154)$$
$$(\langle 9,6,5,2,1\rangle^* - \langle 9,7,4,3\rangle + \langle 11,9,3\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{27} \geq a_{21} \quad (155)$$
$$(\langle 9,7,4,3\rangle - \langle 9,6,5,2,1\rangle^* + \langle 9,8,6\rangle^*) \uparrow^{(5,3)} S_{24}\ hence\ a_{21} \geq a_{27}, \therefore a_{21} = a_{27} \quad (156)$$

then degree$Y_1, Y_2 \equiv 0\ mod\ 7^3$ only when $Y_1 + Y_2 = k_{15}$, so $k_{15} = d_{41} + d_{42}$. When there are 123 columns and $\langle 13,8,3\rangle \neq \langle 13,8,3\rangle'$ on $(7, \alpha)$-regular classes, $k_{10} = d_{27} + d_{28}$. As a result, table 15 represents the decomposition matrix for block B_1.

4. Conclusions

If the field characteristic is prime, the decomposition matrix for the spin characters of the symmetric group is connected between the irreducible spin characters and the irreducible modular spin characters. We had to conduct multiple studies in order to have enough data to discover new properties and theorems because there isn't a general formula for studying the topic, especially when we prove the field and the change of groups. This is what previous researchers did when they looked at the division matrix at the field where the characteristic is 0. We provided 156 equations in this study, including equations Nos. 1, 15, and 16, which were discovered by comparing earlier research and connecting them to a relationship between regular classes (p, α)-regular classes More than 50 equations have been condensed using these equations, also we used maple programming to see all correct probabilities. We also encountered novel issues caused by defect three matrices, which were resolved in cases 1, 2, 3 and 4. This opens the door for a thorough investigation that will first examine matrices of defect four type, then examine irreducible modular spin characters, and lastly classify groups.

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6. Appendix (Maple Programming)

Let the approximation matrix for the block to the spin characters of S_{24} for field has characteristic $p = 7$

The degree	Spin characters	Approximation matrix	
A_1	β_1	a_1	
A_1	$\beta_1{}'$		a_1
A_2	$\beta_2{}^*$	a_2	a_2
.	.	.	.
.	.	.	.
A_i	β_i	a_i	
A_i	$\beta_i{}'$		a_i
		Y_1	Y_2

$a_i \in \mathbb{N}, \forall i$. To find discuss all probabilities such that the degree $Y_1, Y_2 \equiv 0 \ mod \ 7^3$.

> P:=p^3;
D1:=A_1;
D2:=A_2;
.
.
Di:=A_i;
S:=0;
j:=1;
for a_1 from 0 to n_1 do
for a_2 from 0 to n_2 do
.

```
.
for a_i from 0 to n_i do
S:= D1*A_1+ D2*A_2+…+ Di*A_i;
G:=modp(S,P);
if G=0 then
print(j,'a_1'=a_1,'a_2'=a_2,…,'a_i'=a_i);
j:=j+1;
fi;
S:=S;
od;
od;
.          } i index
.
od;
```

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مصفوفة التجزئة للمشخصات الاسقاطية لـ S_{24} مقياس $p = 7$

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الكلمات المفتاحية

التمثيلات والمشخصات المعيارية، المشخصات المعيارية غير القابلة للتحليل، المركزية، الحقول

الملخص

في هذا العمل، نحسب مصفوفة التجزئة للمشخصات الاسقاطية للزمرة التناظرية S_{24}، التي تربط المشخصات الاسقاطية غير القابلة للتحليل بالمشخصات الاسقاطية المعيارية غير القابلة للتحليل، عندما يكون مميز الحقل 7. الطريقه المستخدمة في هذا العمل هي (r، $\bar{r}$) – المستحث لانشاء مشخصات إسقاطيه لـ S_{24} من خلال المشخصات الإسقاطيه لـ S_{23}. واستخدام برنامج لرؤية كل الأعمدة الممكنة ثم اختيار الأعمدة الصحيحة. الهدف من هذا البحث هو تمهيد الطريق لإيجاد العلاقات العامة والنظريات لدراسة المشخصات المعيارية غير القابلة للاختزال وأخيراً لتصنيف الزمر.

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