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Discrete WRM using Tau and Galerkin Techniques to Solve Multi-Higher Fractional Order Integro-Differential Equations of the Fredholm type

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ABSTRACT

The goal of this article is to create and implement novel approaches for utilizing shifted Legendre and Chebyshev polynomials to solve multi-fractional order Fredholm integro-differential equations (FIDEs). The Tau and Galerkin methods will be used in conjunction with the discrete weighted residual approach. The Clenshaw-Curtis quadrature formula will be employed computationally to evaluate the integral terms. In order to identify orthogonal coefficients for approximate solutions, this study uses an operational matrix to transform FIDEs of the fractional orders into a system of linear algebraic equations. In order to obtain approximate solutions for the equation, this procedure enables the development of algorithms for each method. Additionally, the technique's validity, applicability, and similarities with previous results are illustrated and contrasted using numerical examples. Most applications are run on a computer with MATLAB V.9.7 installed.

1. Introduction

This article investigates the family of equations known as Fredholm integro-differential equations (FIFDEs) with multi-fractional orders in the Caputo sense with variable coefficients. They take the following general form:

$$ {}_a^C D_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} P_i(t)\, {}_a^C D_t^{\alpha_{n-i}} u(t) + P_n(t) u(t) = f(t) + \lambda \sum_{j=0}^{m} \int_a^b \mathcal{K}_j(t,s)\, {}_a^C D_s^{\beta_{m-j}} u(s) ds, \quad a \le t \le b \qquad (1) $$

subject to the boundary condition:

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$$\sum_{\ell=1}^{\mu}\left\{g_{k\ell}\,u^{(\ell-1)}(a)+h_{k\ell}\,u^{(\ell-1)}(b)\right\}=C_k\,,\qquad k=1,2,\dots,\mu \tag{2}$$

where $g_{k\ell}, h_{k\ell}$ and $C_k \in \mathbb{R}$ for all $k = 1,2,\dots,\mu$ and $\ell = 1,2,\cdots,\mu$; $\mu = \max\{\lceil\alpha_n\rceil, \lceil\beta_m\rceil\}$. The fractional orders $\{\alpha_i , \beta_j\} \in \mathbb{R}^+ \cup \{0\}$ such that with property: $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > \alpha_0 = 0$ and $\beta_m > \beta_{m-1} > \cdots > \beta_1 > \beta_0 = 0$ and $m_{\alpha_i} - 1 < \alpha_i \le m_{\alpha_i}$ and $m_{\beta_j} - 1 < \beta_j \le m_{\beta_j}$ where $m_{\alpha_i} = \lceil\alpha_i\rceil$ and $m_{\beta_j} = \lceil\beta_j\rceil$ for all $i = 1,2,\dots,n$ and $j = 1,2,\dots,m$. Since $\rho = \{\alpha_{n-i}, \beta_{m-j}\}$ for all $i = 0,1,\dots,n-1$ and $j = 0,1,\dots,m-1$, $n, m \in \mathbb{Z}^+$, the ${}_{a}^{C}D_t^{\rho}$ represents the ρ-Caputo fractional differential operator of the real-valued function $u(t)$ on the closed bounded interval $[a, b]$. The unknown function to be found in equation (1) is u. Additionally, the functions $f, P_i \in C([a, b], \mathbb{R})$, $\mathcal{K}_j \in C(\Xi, \mathbb{R})$, and the $\Xi = \{(t, s): a \le t < s \le b\}$ represented the known continuous function for all $i = 1,2,\dots,n$ and $j = 0,1,\dots,m$. A scalar parameter is the λ.

The integro-differential equation of arbitrary order is an important topic in modern mathematics. These kinds of equations are found in many fields of bioengineering, physical phenomena, and applied mathematics [1, 2]. One major kind that appears in many different application areas is Fredholm. In order to solve fractional differential, fractional integral, and integro-fractional differential equations, numerous researchers and authors have taken these approaches into consideration [3-9]. This motivates us to think about their efficient numerical solution in more recent times. One of the most commonly utilized minimization approaches for determining all the expansion parameters is the weighted residual method (WRM).

The key elements of the WRM are the trial functions and the test functions. The trial functions are used as the basis functions for a truncated series expansion solution. The test functions are used to ensure that the equation is satisfied as closely as possible by this truncated series expansion; this is achieved by minimizing the residual error [10, 11]. The trial functions are usually smooth functions that are supported in the complete domain $[a, b]$. There are many choices possible, in particular, trigonometric functions and orthogonal polynomials. On the other side, the choice of the test functions distinguishes between the most commonly used schemes, namely, the collocation, sub-domain, moment, and least-square methods. These classes of techniques are used to build some algorithms for computing problems such as ordinary, partial, and fractional differentials, integrals, and integro-differential equations [12-14]. In this paper, we extend this technique to further deal with problem (1) with given boundary conditions (2).

The present work is arranged as follows:

- Section 2 outlines the basic definitions, supporting lemmas, and principal properties related to fractional integrals and derivatives.
- Section 3 basic properties of shifted orthogonal polynomials (Legendre and Chebyshev) is given.
- Section 4 preliminary of the Clenshaw-Curtis quadrature rule is given.
- Section 5 we explain how to create the new method by combining WRM with shifted orthogonal polynomials and applying it to FIFDEs, detailing the numerical techniques and algorithms for the Tau and Galerkin techniques.
- Section 6 focuses on numerical illustrative examples. As well as, offers a comparison between our algorithms and existing approaches to assess their effectiveness.
- Finally, Section 7 presents a discussion of the numerical results and their implications.

2. Fractional Derivatives

Fractional derivatives' fundamental definitions and characteristics are covered in this section. The explanation of a number of basic ideas and lemmas used in this study was also provided.

Definition 1 ([15]). A real-valued function u, defined on a closed bounded interval $[a, b]$, belongs to the space $C_d[a, b], d \in \mathbb{R}$ if there is a real number $e > d$ such that $u(t) = (t-a)^e u_*(t)$, with $u_* \in C[a, b]$. If and only if it's m-th derivative $u^{(m)}(t)$ similarly belongs to $C_d[a, b], m \in \mathbb{Z}^+$, it is also considered to belong to the space $C_d^m[a, b]$.

Definition 2 ([1, 16]). For a function $u \in C_d[a,b], d \geq -1$, the left-sided Riemann-Liouville (R-L) fractional integral of order ϑ is defined as

$$ {}_aJ_t^{\vartheta}u(t) = \frac{1}{\Gamma(\vartheta)}\int_a^t (t-\xi)^{\vartheta-1}\, u(\xi)d\xi, \qquad \vartheta \in \mathbb{R}^+, \qquad a \leq t \leq b $$

Here, $\Gamma(.)$ denotes the gamma function, and for $\vartheta = 0$, we have the Riemann-Liouville identity operator, ${}_aJ_t^0u(t) = u(t)$, and has the semi-commutative property: ${}_aJ_t^{\vartheta_1}\,{}_aJ_t^{\vartheta_2}u(t) = {}_aJ_t^{\vartheta_1+\vartheta_2}u(t) = {}_aJ_t^{\vartheta_2}\,{}_aJ_t^{\vartheta_1}u(t)$ for all $\vartheta_1, \vartheta_2 \geq 0$. While the R-L fractional power function on a closed interval $[a,b]$ formulated as:

$$ {}_aJ_t^{\vartheta}(t-a)^{\kappa} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\vartheta+1)}(t-a)^{\kappa+\vartheta}, \qquad \vartheta \in \mathbb{R}^+, \qquad \kappa > -1 $$

The most commonly used definitions are the Riemann-Liouville and Caputo formulas. But the Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional equations. Therefore, the Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

Definition 3 ([16, 17]). The operator ${}_a^CD_t^{\vartheta}u(t)$, for a function $u \in C_{-1}^{\lceil\vartheta\rceil}[a,b]$ of order $\vartheta \geq 0$ and $t > a$, defined as

$$ {}_a^CD_t^{\vartheta}u(t) = {}_aJ_t^{\lceil\vartheta\rceil-\vartheta}D_t^{\lceil\vartheta\rceil}u(t) = \frac{1}{\Gamma(\lceil\vartheta\rceil-\vartheta)}\int_a^t (t-\xi)^{\lceil\vartheta\rceil-\vartheta-1}\frac{d^{\lceil\vartheta\rceil}u(\xi)}{d\xi^{\lceil\vartheta\rceil}}d\xi \qquad (3) $$

is called the Caputo fractional differential operator of order ϑ. In the Caputo manner derivative for $\vartheta = 0$, we have the Caputo identity derivative operator, ${}_a^CD_t^0u(t) = u(t)$. The following properties hold:

- ${}_a^CD_t^{\vartheta}\mathcal{A} = 0$; $\mathcal{A}$ is any constant;($\vartheta \geq 0, \vartheta \notin \mathbb{N}$).
- ${}_a^CD_t^{\vartheta}\ {}_aJ_t^{\vartheta}u(t) = u(t)$, for $\lceil\vartheta\rceil - 1 < \vartheta \leq \lceil\vartheta\rceil,\ a \leq t \leq b$.
- ${}_aJ_t^{\vartheta}\,{}_a^CD_t^{\vartheta}u(t) = u(t) - \sum_{\kappa=0}^{\lceil\vartheta\rceil-1}\frac{u^{(\kappa)}(a)}{\kappa!}(t-a)^{\kappa}$, for $\lceil\vartheta\rceil - 1 < \vartheta \leq \lceil\vartheta\rceil$.

Lemma 1([2, 17]). The function $u(t) = (t-a)^{\kappa}$, for $\kappa \geq 0$, has a Caputo derivative of order $\vartheta \geq 0$, which is formed as: For $\kappa \in \{0,1,2,\cdots,\lceil\vartheta\rceil-1\}$: ${}_a^CD_t^{\vartheta}u(t) = 0$ and for $\kappa \in \mathbb{N}$ and $\kappa \geq \lceil\vartheta\rceil$ or $\kappa \notin \mathbb{N}$ and $\kappa > \lceil\vartheta\rceil - 1$:

$$ {}_a^CD_t^{\vartheta}u(t) = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\vartheta+1)}(t-a)^{\kappa-\vartheta} \qquad (4) $$

Lemma 2 ([17]). Let say $\vartheta \geq 0$, $\vartheta \notin \mathbb{N}$. and for $u \in C_{-1}^{\lceil\vartheta\rceil}[a,b]$. Then $[{}_a^CD_t^{\vartheta}u(t)]_{t=a} = 0$, that is $\lim_{t\to a}[{}_a^CD_t^{\vartheta}u(t)] = 0$, and the Caputo fractional derivative ${}_a^CD_t^{\vartheta}u(t)$ is continuous on $[a,b]$.

3. Shifted Orthogonal Polynomials (SOPs):

3.1 The Properties of the Shifted Legendre Polynomials (SLPs) ([18, 19]).

The set of Legendre polynomials of degree r in the variable x over the interval $[-1,1]$ are denoted by $\mathcal{P}_r(x)$ then may be generated by the recurrence formula:

$$ \mathcal{P}_{r+1}(x) = \frac{2r+1}{r+1}x\,\mathcal{P}_r(x) - \frac{r}{r+1}\mathcal{P}_{r-1}(x)\,, r = 1,2,\ldots \qquad \mathcal{P}_0(x) = 1\,, \mathcal{P}_1(x) = x $$

The shifted domain can be obtained by introducing the following linear transformation:

$$ \left. x = 2\left(\frac{t-a}{b-a}\right) - 1, a \leq t \leq b \quad \text{and} \quad t = \frac{b-a}{2}x + \frac{b+a}{2}, -1 \leq x \leq 1 \right\} \qquad (5) $$

Legendre polynomials, also known as r -shifted Legendre polynomials (SLPs) in $t, \mathcal{P}_r^*(t)$, are defined on the closed bounded interval $[a,b]$ and are produced by the recurrence formula

$$\left.\begin{aligned}\mathcal{P}_r^*(t) &= \frac{2r+1}{r+1}\left(\frac{2t-a-b}{b-a}\right)\mathcal{P}_r^*(t) - \frac{r}{r+1}\mathcal{P}_{r-1}^*(t), \qquad r = 1,2,\dots \\ \mathcal{P}_0^*(t) &= 1\ , \qquad \mathcal{P}_1^*(t) = \frac{2t-a-b}{b-a}\end{aligned}\right\} \tag{6}$$

$\mathcal{P}_r^*(t)$, the explicit analytical series form of the SLPs of degree r, can be expressed as follows:

$$\mathcal{P}_r^*(t) = \frac{1}{2^r}\sum_{\kappa=0}^{\lfloor r/2\rfloor}(-1)^\kappa\binom{r}{\kappa}\binom{2r-2\kappa}{r}\left[2\left(\frac{t-a}{b-a}\right)-1\right]^{r-2\kappa}, \qquad r \geq 1 \tag{7}$$

and this in turn enables one to get

$$\mathcal{P}_d^*(a) = (-1)^d \text{ and } \mathcal{P}_d^*(b) = 1 \tag{8}$$

$$\left.\begin{aligned}\frac{\mathrm{d}\mathcal{P}_d^*(a)}{\mathrm{d}t} &= \frac{1}{b-a}(-1)^{d-1}d(d+1) \\ \frac{\mathrm{d}\mathcal{P}_d^*(b)}{\mathrm{d}t} &= \frac{1}{b-a}d(d+1)\end{aligned}\right\} \tag{9}$$

$$\frac{\mathrm{d}^2\mathcal{P}_d^*(t)}{\mathrm{d}t^2} = \begin{cases}\text{at } t = a\ , \dfrac{1}{2(b-a)^2}(-1)^d(d-1)d(d+1)(d+2) \\ \text{at } t = b\ , \dfrac{1}{2(b-a)^2}(d-1)d(d+1)(d+2)\end{cases} \tag{10}$$

Lemma 3 ([19]). The $n(\in \mathbb{Z}^+)$-derivative of the r-SLPs on any closed bounded interval $[a,b], (a < b)$, are expressed as:

$$\frac{\mathrm{d}^n\mathcal{P}_r^*(t)}{\mathrm{d}t^n} = \left\{\begin{array}{ll}\frac{1}{2^r}\sum_{\kappa=0}^{\left\lfloor\frac{r-n}{2}\right\rfloor}\frac{(-1)^\kappa}{\kappa!(r-\kappa)!}\frac{(2r-2\kappa)!}{(r-2\kappa-n)!}\left[\frac{2}{b-a}\right]^n\left[2\left(\frac{t-a}{b-a}\right)-1\right]^{r-2\kappa-n} & \text{if } r > n \\ \frac{(2r)!}{2^r r!}\left[\frac{2}{b-a}\right]^r & \text{if } r = n \\ 0 & \text{if } r < n\end{array}\right\} \tag{11}$$

Lemma 4 ([19]). The ϑ-Caputo fractional derivative for order ϑ where $m-1 < \vartheta < m, m = \lceil\vartheta\rceil$, of SLPs for degree $r(\geq 1)$, $\mathcal{P}_r^*(t) = \mathcal{P}_r\left[2\left(\frac{t-a}{b-a}\right)-1\right]$, on a closed bounded interval $[a,b]$ can be formulated:

$${}_a^C D_t^\vartheta \mathcal{P}_r^*(t) = \frac{1}{2^r}\left(\frac{2}{b-a}\right)^m (t-a)^{m-\vartheta}\sum_{\kappa=0}^{\lfloor r/2\rfloor}(-1)^\kappa\frac{\Gamma(2r-2\kappa+1)}{\Gamma(\kappa+1)\Gamma(r-\kappa+1)}\, M_\vartheta^L(t; r,\kappa,m) \tag{12}$$

where

$$M_\vartheta^L(t; r,\kappa,m) = \left\{\begin{array}{ll}0 & , m > r-2\kappa \\ \frac{1}{\Gamma(m-\vartheta+1)} & , m = r-2\kappa \\ \sum_{\ell=0}^{r-2\kappa-m}\frac{(-1)^{\ell+r-2\kappa-m}}{\Gamma(\ell+m-\vartheta+1)\Gamma(r-2\kappa-m-\ell+1)}\left[2\left(\frac{t-a}{b-a}\right)\right]^\ell & , m < r-2\kappa\end{array}\right\} \tag{13}$$

3.2 The Properties of the Shifted Chebyshev Polynomials (SCPs) ([17, 18, 19]).

The set of Chebyshev polynomials of degree ($r \in \mathbb{Z}^+$), denoted by $\mathcal{T}_r(x)$ on the interval $x \in [-1,1]$, are the sequence of orthogonal polynomials with respect to the weighted function $(1-x^2)^{-1/2}$. A simple fundamental recurrence formula:

$$\mathcal{T}_{r+1}(x) = 2x\mathcal{T}_r(x) - \mathcal{T}_{r-1}(x), r = 1,2,\dots \ \mathcal{T}_0(x) = 1, \mathcal{T}_1(x) = x$$

To obtain the r-shifted Chebyshev polynomials (SCPs) in t over the bounded closed interval $[a,b]$ it is necessary to shift the defining domain by using equation (5), which yields:

$$\left.\begin{aligned} \mathcal{T}^*_{r+1}(t) &= 2\left[\frac{2t-a-b}{b-a}\right]\mathcal{T}^*_r(t) - \mathcal{T}^*_{r-1}(t), \qquad r = 1,2,\dots \\ \mathcal{T}^*_0(t) &= 1\,, \qquad \mathcal{T}^*_1(t) = \frac{2t-a-b}{b-a} \end{aligned}\right\} \tag{14}$$

Conversely, the series sums can also be used to express the r -SCPs:

$$\mathcal{T}^*_r(t) = \frac{r}{2}\sum_{\kappa=0}^{\left\lfloor\frac{r}{2}\right\rfloor}\frac{(-1)^\kappa 2^{r-2\kappa}}{r-\kappa}\binom{r-\kappa}{\kappa}\left[2\left(\frac{t-a}{b-a}\right)-1\right]^{r-2\kappa}, \qquad r \geq 1$$

and this, in turn, enables one to get:

$$\mathcal{T}^*_d(a) = (-1)^d \text{ and } \mathcal{T}^*_d(b) = 1 \tag{15}$$

for all $\ell = 1,2,3\ldots$ we have

$$\left.\begin{aligned} \left.\frac{\mathrm{d}^\ell \mathcal{T}^*_d(t)}{\mathrm{d}t^\ell}\right|_{t=a} &= \left(\frac{2}{b-a}\right)^\ell (-1)^{d+\ell}\prod_{k=0}^{\ell-1}\frac{d^2-k^2}{2k+1} \\ \left.\frac{\mathrm{d}^\ell \mathcal{T}^*_d(t)}{\mathrm{d}t^\ell}\right|_{t=b} &= \left(\frac{2}{b-a}\right)^\ell \prod_{k=0}^{\ell-1}\frac{d^2-k^2}{2k+1} \end{aligned}\right\} \tag{16}$$

Lemma 5 ([19]). The $n(\in \mathbb{Z}^+)$ -derivative of the r -SCPs on any closed bounded interval $[a,b], (a < b)$, are expressed as:

$$\frac{\mathrm{d}^n \mathcal{T}^*_r(t)}{\mathrm{d}t^n} = \left\{\begin{array}{ll} \frac{r}{2}\sum_{\kappa=0}^{\left\lfloor\frac{r-n}{2}\right\rfloor}(-1)^\kappa 2^{r-2\kappa}\frac{(r-\kappa-1)!}{\kappa!(r-2\kappa-n)!}\left[\frac{2}{b-a}\right]^n\left[2\left(\frac{t-a}{b-a}\right)-1\right]^{r-2\kappa-n} & \text{if } r > n \\ 2^{r-1}(r!)\left[\frac{2}{b-a}\right]^r & \text{if } r = n \\ 0 & \text{if } r < n \end{array}\right\} \tag{17}$$

Lemma 6 ([19]). The ϑ-Caputo fractional derivative for order ϑ where $m-1 < \vartheta < m, m = \lceil\vartheta\rceil$, of SCPs for degree $r(\geq 1)$, $\mathcal{T}^*_r(t) = \mathcal{T}_r\left[2\left(\frac{t-a}{b-a}\right)-1\right]$, on a closed bounded interval $[a,b]$ can be formulated:

$${}^C_aD^\vartheta_t \mathcal{T}^*_r(t) = \frac{r}{2}\left(\frac{2}{b-a}\right)^m (t-a)^{m-\vartheta}\sum_{\kappa=0}^{\lfloor r/2\rfloor}(-1)^\kappa\frac{\Gamma(r-\kappa)}{\Gamma(\kappa+1)}2^{r-2\kappa} M^C_\vartheta(t;r,\kappa,m) \tag{18}$$

where

$$M^C_\vartheta(t;r,\kappa,m) = \left\{\begin{array}{ll} 0 & ,m > r-2\kappa \\ \frac{1}{\Gamma(m-\vartheta+1)} & ,m = r-2\kappa \\ \sum_{\ell=0}^{r-2\kappa-m}\frac{(-1)^{\ell+r-2\kappa-m}}{\Gamma(\ell+m-\vartheta+1)\Gamma(r-2\kappa-m-\ell+1)}\left[2\left(\frac{t-a}{b-a}\right)\right]^\ell & ,m < r-2\kappa \end{array}\right\} \tag{19}$$

4. Clenshaw-Curtis Quadrature Formula ([19, 22]).

Various numerical calculations for the solution of integral parts in any equations are based on a numerical quadrature rule. The R-Clenshaw-Curtis quadrature procedure is developed in this section using the relation to present the expansion of the integrand in terms of Chebyshev polynomials and the extreme Chebyshev zeros. That is represented by means of the relation,

$$\int_a^b h(t)dt \cong \frac{b-a}{2}\sum_{k=0}^{R}{}'' w_k h(\tau_k) \tag{20}$$

where the double prime on the summation symbol here and elsewhere indicates that the terms with suffixes $k = 0$ and R are to be halved, and the points $\{\tau_k\}$ are R-shifted Chebyshev collocation points:

$$\eta_k = cos\left(k\frac{\pi}{R}\right)\ , \qquad \tau_k = \frac{b-a}{2}\eta_k + \frac{b+a}{2}$$

$$w_k = \frac{4}{R} \sum_{\ell=0}^{Q}{}'' \mathcal{V}_\ell cos\left(k\ell\frac{\pi}{Q}\right) \;;\; Q \in \mathbb{Z}^+ \quad \text{and} \quad \mathcal{V}_\ell = \begin{cases} 0, & \text{for } \ell \text{ is odd} \\ \dfrac{1}{1-\ell^2}, & \text{for } \ell \text{ is even} \end{cases}$$

5. Solution Technique

In this section, a new algorithm for solving approximately linear multi-higher fractional order integro-fractional differential equations with variable coefficients of Fredholm type, applying discrete weighted residual methods with the use of orthogonal trial functions, has been presented. The starting point of this method is to approximate the solution $u(t)$ of (1) with mixed-condition equations (2) by a finite sum:

$$u(t) \cong \widehat{U}_N(t) = \sum_{d=0}^{N} \hat{u}_d \varphi_d(t) \tag{21}$$

where $\{\varphi_d\}_{d=0}^{N}$ are the N-trial (or basis) functions, and the expansion coefficients $\{\hat{u}_d\}_{d=0}^{N}$ are to be determined for all d. Substituting $\widehat{U}_N$ for u in equation (1) leads to the residual (error) function:

$$E_N\big(t, \widehat{U}\big) = L[U_N(t)] - f(t) \neq 0, \qquad t \in [a, b] \tag{22}$$

where $\widehat{U} = [\hat{u}_0, \hat{u}_1, \dots, \hat{u}_N]$ and $L[\varphi]$ denote a linear operator, which is defined

$$\begin{aligned} L[\varphi] \\ &= {}_a^C D_t^{\alpha_n} \varphi(t) + \sum_{i=1}^{n-1} P_i(t) {}_a^C D_t^{\alpha_{n-i}} \varphi(t) + P_n(t)\varphi(t) \\ &- \lambda \int_a^b \sum_{j=0}^{m} \mathcal{K}_j(t,s) {}_a^C D_s^{\beta_{m-j}} \varphi(s) ds \end{aligned} \tag{23}$$

The function $U_N(t)$ is a suitable solution to the problem (1) and (2) if and only if it makes the residual function $E_N\big(t, \widehat{U}\big)$ as small as possible. The notion of the WRM is to choose the coefficients $\hat{u}_d(d = \overline{0{:}N})$ such that the residual E_N becomes small over a domain, this means to determine the $N + 1$ unknown coefficients $\hat{u}_d$, the method of weighted residuals requires that the residual E_N to be orthogonal with all chosen $N + 1$ test functions $\{\psi_d\}_{d=0}^{N}$ over the domain $[a, b]$. The smallness of E_N is enforced by demanding that

$$\langle \psi_d, E_N(t; \hat{u}_0, \hat{u}_1, \dots, \hat{u}_N) \rangle_{M,\overline{W}} = 0, \qquad \text{for all } d \le N \tag{24}$$

where $\langle .,. \rangle_{M,\overline{W}}$ is the discrete inner product for the set of preselected M-points with associated weights $\overline{W}$, defined by: $\langle \psi, \phi \rangle_{M,\overline{W}} = \sum_{r=0}^{M} \overline{w}_r \psi(t_r) \phi(t_r)$, [20]. Discrete weighted residual approaches are the methodology outlined in equation (24). The selection of trial/test functions is one of the key characteristics that set WRMs apart. The trial functions $\{\varphi_d\}$ that are most frequently utilized are smooth functions, such as orthogonal and trigonometric functions. But we concentrate on two significant orthogonal polynomials that are defined in Section 3, including: $\varphi_d(t) = \mathcal{P}_d^*(t) = \mathcal{P}_d\left[2\left(\frac{t-a}{b-a}\right) - 1\right]$ or $\varphi_d(t) = \mathcal{T}_d^*(t) = \mathcal{T}_d\left[2\left(\frac{t-a}{b-a}\right) - 1\right]$. Here, $\mathcal{P}_d^*$ and $\mathcal{T}_d^*$ are SLPs and SCPs of degree $d (\in \mathbb{Z}^+)$, respectively. Thus, instead of the trial functions, we can take SLPs or SCPs to approximate the solution $u(t)$ of equation (16) by:

$$\widehat{U}_N(t) = \sum\nolimits_{d=0}^{N} \hat{u}_d \mathcal{P}_d\left[2\left(\frac{t-a}{b-a}\right) - 1\right] \tag{25}$$

or

$$\widehat{U}_N(t) = \sum\nolimits_{d=0}^{N} \hat{u}_d \mathcal{T}_d\left[2\left(\frac{t-a}{b-a}\right) - 1\right] \tag{26}$$

So, the residual function (22) becomes:

$$E_N(t, \hat{u}_0, \hat{u}_1, \dots, \hat{u}_N) = \sum\nolimits_{d=0}^{N} \hat{u}_d L[\mathcal{P}_d^*(t)] - f(t) \tag{27}$$

or

$$E_N(t,\hat{u}_0,\hat{u}_1,\dots,\hat{u}_N)=\sum\nolimits_{d=0}^{N}\hat{u}_d L[\mathcal{T}_d^*(t)]-f(t) \tag{28}$$

where L is the linear operator defined at equation (23), and putting SLPs and SCPs to obtain $L[\mathcal{P}_d^*(t)]$ and $L[\mathcal{T}_d^*(t)]$ for equations (27) and (28), respectively. In practice, these integrals will have to be approximated by a numerical integration technique, say Clenshaw-Curtis quadrature formula (see section 4). Thus:

$$\begin{aligned}&L[\mathcal{P}_d^*(t)]\\&={}_a^C D_t^{\alpha_n}\mathcal{P}_d^*(t)+\sum_{i=1}^{n-1}P_i(t)\,{}_a^C D_t^{\alpha_{n-i}}\mathcal{P}_d^*(t)+P_n(t)\mathcal{P}_d^*(t)\\&-\lambda\frac{b-a}{2}\sum_{j=0}^{m}\left\{\sum_{k=0}^{R}{}'' w_k\mathcal{K}_j(t,s_k)\left[{}_a^C D_s^{\beta_{m-j}}\mathcal{P}_d^*(s)\right]_{s=s_k}\right\}\end{aligned} \tag{29}$$

$$\begin{aligned}&L[\mathcal{T}_d^*(t)]\\&={}_a^C D_t^{\alpha_n}\mathcal{T}_d^*(t)+\sum_{i=1}^{n-1}P_i(t)\,{}_a^C D_t^{\alpha_{n-i}}\mathcal{T}_d^*(t)+P_n(t)\mathcal{T}_d^*(t)\\&-\lambda\frac{b-a}{2}\sum_{j=0}^{m}\left\{\sum_{k=0}^{R}{}'' w_k\mathcal{K}_j(t,s_k)\left[{}_a^C D_s^{\beta_{m-j}}\mathcal{T}_d^*(s)\right]_{s=s_k}\right\}\end{aligned} \tag{30}$$

Where the double prime on the summation symbol indicates that the terms with suffixes $k=0$ and R to be halved, the points $\{s_k\}$ are Chebyshev collocation points, and where Q is any given positive integer number, as in section 4. Thus:

$$\eta_k=\cos\left(\frac{k\pi}{R}\right),\quad s_k=\frac{b-a}{2}\eta_k+\frac{b+a}{2},\quad w_k=\frac{4}{R}\sum_{\substack{\ell=0\\ even}}^{Q}{}''\frac{1}{1-\ell^2}\cos\left(k\ell\frac{\pi}{Q}\right)$$

Now, for test functions $\{\psi_d\}$, there exist various methods to choose them. Here, we mention the most common approaches, namely, the collocation, sub-domain, moment, and least squares methods, to determine the arbitrary parameters $\hat{u}_d (d=0,1,\dots,N)$ according to the choices of test functions $\{\psi_d\}$ in the equation (24).

5.1 Tau-Method (TM)

In this technique, the test function ψ_d are chosen to be identical to the trial functions φ_ℓ, that is, $\psi_\ell(t)=\varphi_\ell(t)$, for all $\ell=0,1,\dots,N$; [19, 20]. The application of orthogonal polynomials to the WR-Tau method is to let $\varphi_\ell(t)=\mathcal{P}_\ell^*(t)$ or $\mathcal{T}_\ell^*(t)$, especially in our work. So, this choice implies the residual equation (19) to equation (26):

$$\sum_{r=0}^{M}\bar{w}_r\,\varphi_\ell(t_r)E_N(t_r,\hat{u}_0,\hat{u}_1,\dots,\hat{u}_N)=0,\qquad \ell\le N \tag{31}$$

In the present cases, the selection of $(M+1)$-points with the associated values of weights $\{t_r,\bar{w}_r\}_{r=0}^{M}$ are defined for SLPs in equation (32) and for SCPs in (33). The values of the weights $\{\bar{w}_r\}$ and collocation points $\{t_r\}$ can be taken for the two usual cases, depending on choosing the trial functions. That is:

i. SLPs $\{\mathcal{P}_d^*\}$ instead of trial functions $\{\varphi_d\}$. We choose the Gauss-Legendre sets $\{t_r,\bar{w}_r\}$: τ_r are the nodes of $\mathcal{P}_{M+1}(t)=0$; and

$$\left.t_r=\frac{b-a}{2}\tau_r+\frac{b+a}{2}\quad\text{and}\quad\bar{w}_r=\frac{2}{(1-\tau_r^2)[P'_{M+1}(\tau_r)]^2}\right\} \tag{32}$$

ii. Where taking SCPs $\{\mathcal{T}_d^*\}$ instead of trial functions $\{\varphi_d\}$. We choose the Gauss-Chebyshev-Labotto sets $\{t_r,\bar{w}_r\}$:

$$\tau_r=-cos\left(\frac{r\pi}{M}\right);\ t_r=\frac{b-a}{2}\tau_r+\frac{b+a}{2}\quad\text{and}\quad\bar{w}_r=\begin{Bmatrix}\pi/2M, & \text{for } r=0,M\\ \pi/M, & \text{else}\end{Bmatrix} \tag{33}$$

We get $(N+1)$-linear algebraic equations from equation (31) by calling the equations (27 with 29) or (28 with 30):

$$\sum_{d=0}^{N} \hat{u}_d \left\{ \sum_{r=0}^{M} \bar{w}_r \varphi_\ell(t_r) \left[\sum_{i=0}^{n} P_i(t_r) \left[{}_a^C D_t^{\alpha_{n-i}} \varphi_d(t) \right]_{t=t_r} - \frac{\lambda(b-a)}{2} \sum_{j=0}^{m} \sum_{k=0}^{R}{}'' w_k \mathcal{K}_j(t_r, s_k) \left[{}_a^C D_s^{\beta_{m-j}} \varphi_d(s) \right]_{s=s_k} \right] \right\} = \sum_{r=0}^{M} \bar{w}_r \varphi_\ell(t_r) f(t_r), \qquad \ell \le N \tag{34}$$

For all $\ell = 0,1,\dots,N$ and the set $\{s_k, w_k\}_{k=0}^{R}$ are selected from the R-Clenshaw-Curtis quadrature procedure in section 4, as in equation (35).

$$\left. \begin{aligned} \eta_k = -\cos\left(\frac{k\pi}{R}\right) \quad ; \quad s_k = \frac{b-a}{2}\eta_r + \frac{b+a}{2} \\ w_k = \frac{4}{R} \sum_{\substack{\ell=0 \\ even}}^{Q}{}'' \frac{1}{1-\ell^2} \cos\left(k\ell\frac{\pi}{Q}\right) \end{aligned} \right\} \tag{35}$$

However, the boundary conditions (2) include the μ-linear equations, which can be expressed as follows:

$$\sum_{d=0}^{N} \hat{u}_d \Omega_{kd} = \mathcal{C}_k, \qquad \text{for all } k = 1,2,\dots,\mu \tag{36}$$

where Ω_{kd}, for each k and d, are constant values which is denoted by:

$$\Omega_{kd} = \sum_{\ell=1}^{\mu} g_{k\ell} \left[\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}t^{\ell-1}} \varphi_d(t) \right]_{t=a} + h_{k\ell} \left[\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}t^{\ell-1}} \varphi_d(t) \right]_{t=b}$$

In matrix form, μ-equations (36) that give:

$$\mathcal{B}_\mu \hat{U}^T = \mathcal{C}_\mu \tag{37}$$

where $\mathcal{B}_\mu$ is a matrix of dimension $\mu \times (N+1)$, defines:

$$\mathcal{B}_\mu = [\Omega_{kd}] = \begin{bmatrix} \Omega_{10} & \Omega_{11} & \cdots & \Omega_{1N} \\ \Omega_{20} & \Omega_{21} & \cdots & \Omega_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{\mu 0} & \Omega_{\mu 1} & \cdots & \Omega_{\mu N} \end{bmatrix}_{\mu \times (N+1)}$$

with $\mathcal{C}_\mu = [\mathcal{C}_1 \quad \mathcal{C}_2 \quad \cdots \quad \mathcal{C}_\mu]^T$ and $\hat{U} = [\hat{u}_0 \quad \hat{u}_1 \quad \cdots \quad \hat{u}_N]^T$.

These equations (34) and the μ-boundary equation (36) can be combined to create a set of $(N+\mu+1)$-linear algebraic equations, which, when solved, yield unknown Tau-parameters $\hat{u}_d$. In matrix form, rewrite equation (34) as follows:

$$\mathcal{H}_{\mathcal{T}}(\varphi) \hat{U}^T = \mathcal{F}_{\mathcal{T}} \tag{38}$$

Define the row-block vector $\mathcal{H}_{\mathcal{T}}(\varphi)$ as follows:

$$\mathcal{H}_{\mathcal{T}}(\varphi) = [G_0(\varphi) \quad G_1(\varphi) \quad \cdots \quad G_N(\varphi)] \quad \text{and} \quad \mathcal{F}_{\mathcal{T}} = \mathcal{T}_t \bar{W} \mathcal{F}$$

And, for all $d = 0,1,\dots,N$

$$G_d(\varphi) = \mathcal{T}_t(\varphi) \bar{W} \left[\sum_{i=0}^{n} \bar{P}_i M_{n,i}^{\alpha}(\varphi_d) - \frac{\lambda(b-a)}{2} \sum_{j=0}^{m} \boldsymbol{\mathcal{K}_j} W R_{m,j}^{\beta}(\varphi_d) \right]$$

and

$$\mathcal{T}_t(\varphi) = \begin{bmatrix} \varphi_0(t_0) & \varphi_0(t_1) & \cdots & \varphi_0(t_M) \\ \varphi_1(t_0) & \varphi_1(t_1) & \cdots & \varphi_1(t_M) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_N(t_0) & \varphi_N(t_1) & \cdots & \varphi_N(t_M) \end{bmatrix}_{(N+1)\times(M+1)}$$

and $\mathcal{F} = \left([f(t_0) \quad f(t_1) \quad \cdots \quad f(t_M)]_{1\times(M+1)}\right)^T, \bar{P}_i = \mathrm{diag}[P_i(t_0) \quad P_i(t_1) \quad \cdots \quad P_i(t_M)]$, $\bar{W} = \mathrm{diag}[\bar{w}_0 \quad \bar{w}_1 \quad \cdots \quad \bar{w}_M]$, $W = \mathrm{diag}\left[\frac{1}{2}w_0 \quad w_1 \quad \cdots \quad w_{R-1} \quad \frac{1}{2}w_R\right]$ and

$$M_{n,i}^{\alpha}(\varphi_d) = \left(\left[{}_a^C D_t^{\alpha_{n-i}}\varphi_d(t_0) \quad {}_a^C D_t^{\alpha_{n-i}}\varphi_d(t_1) \quad \cdots \quad {}_a^C D_t^{\alpha_{n-i}}\varphi_d(t_M)\right]_{1\times(M+1)}\right)^T$$

with

$$R_{m,j}^{\beta}(\varphi_d) = \left(\left[{}_a^C D_s^{\beta_{m-j}}\varphi_d(s_0) \quad {}_a^C D_s^{\beta_{m-j}}\varphi_d(s_1) \quad \cdots \quad {}_a^C D_s^{\beta_{m-j}}\varphi_d(s_R)\right]_{1\times(R+1)}\right)^T$$

while for all $r = \overline{0:M}$ and $k = \overline{0:R}$, for $j = 0,1,\dots,m$:

$$\boldsymbol{\mathcal{K}}_j = \left[\mathcal{K}_{rk}^j\right] = \begin{bmatrix} \mathcal{K}_{0,0}^j & \mathcal{K}_{0,1}^j & \cdots & \mathcal{K}_{0,R}^j \\ \mathcal{K}_{1,0}^j & \mathcal{K}_{1,1}^j & \cdots & \mathcal{K}_{1,R}^j \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{K}_{M,0}^j & \mathcal{K}_{M,1}^j & \cdots & \mathcal{K}_{M,R}^j \end{bmatrix}_{(M+1)\times(R+1)}$$

We generate the matrix of dimension $(N+\mu+1)\times(N+1)$ and $(N+\mu+1)\times 1$ by adding the block-condition matrices $\mathcal{B}_\mu$ and $\mathcal{C}_\mu$ as in equation (37) into matrices (34):

$$\begin{bmatrix} \mathcal{H}_{\mathcal{T}}(\varphi) \\ \mathcal{B}_\mu \end{bmatrix} \hat{U}^T = \begin{bmatrix} \mathcal{F}_{\mathcal{T}} \\ \mathcal{C}_\mu \end{bmatrix}$$

After applying iterative approaches to such a linear system, the square matrix for determining the Tau-coefficients $\hat{u}_d$ is obtained by multiplying both sides by the transformation of the first matrix:

$$\begin{bmatrix} \mathcal{H}_{\mathcal{T}}(\varphi) \\ \mathcal{B}_\mu \end{bmatrix}^T \begin{bmatrix} \mathcal{H}_{\mathcal{T}}(\varphi) \\ \mathcal{B}_\mu \end{bmatrix} \hat{U}^T = \begin{bmatrix} \mathcal{H}_{\mathcal{T}}(\varphi) \\ \mathcal{B}_\mu \end{bmatrix}^T \begin{bmatrix} \mathcal{F}_{\mathcal{T}} \\ \mathcal{C}_\mu \end{bmatrix} \tag{39}$$

Then substitute the values of $\hat{u}_d$'s into expansion (21), the approximate solution is obtained for FIFDE (1) and (2). Here, instead of the trial functions φ_d's taking SLPs $\{\mathcal{P}_d^*\}$ and SCPs $\{\mathcal{T}_d^*\}$ to obtain a good approximate solution $\hat{U}(t)$ to $u(t)$ of problem FIFDE's, so that for evaluating the fractional parts $M_{n,i}^{\alpha}(\mathcal{P}_d^* \text{ or } \mathcal{T}_d^*)$ and $R_{m,j}^{\beta}(\mathcal{P}_d^* \text{ or } \mathcal{T}_d^*)$ at all points $t_r(r = \overline{0:N})$, applying Lemma 4 for SLPs and Lemma 6 for SCPs, respectively. Also, for differential values Ω_{kd} elements in matrix (36) applying Lemma 3 for Legendre and Lemma 5 for Chebyshev orthogonal polynomials, respectively. For these stages, we had written the following algorithm.

The Algorithm (ATM). The approximate solution of multi-higher IFDEs of Fredholm type (1) and boundary conditions (2) by applying the discrete WR-Tau method with orthogonal polynomials SLPs and SCPs can be summarized in the following stages:

Step 1.

- Input N-number of approximate terms, (R,Q)-number of terms in the quadrature integration formula and M-number of points in the discrete inner product, which are the Gauss-Legendre or Gauss-Chebyshev-Labotto collocation points.
- Set $t_r = \frac{b-a}{2}\tau_r + \frac{b+a}{2}$ where τ_r 's are the $M+1$ roots of $\mathcal{P}_{M+1}(t)$ with $\bar{w}_r = 2/\{(1-\tau_r^2)[P'_{M+1}(\tau_r)]^2\}$ or $\tau_r = -\cos(r\pi/M)$ with $\bar{w}_r = \pi/2M$ for $r = 0, M$, and otherwise $\bar{w}_r = \pi/M$.
- Put $s_k = \frac{b-a}{2}\eta_k + \frac{b+a}{2}$ where $\eta_k = \cos\left(\frac{k\pi}{R}\right)$, for each $k = \overline{0:R}$ and given Q , with

$$w_k = \frac{4}{R} \sum_{\substack{\ell=0 \\ even}}^{Q\ ''} \frac{1}{1-\ell^2} cos\left(k\ell\frac{\pi}{Q}\right)$$

Step 2. For all $d = 0,1,\dots,N$

- Evaluate $M_{n,i}^{\alpha}(\varphi_d)$ and $R_{m,j}^{\beta}(\varphi_d)$ for all fractional orders $\{\alpha_i\}_{i=0}^n$ and $\{\beta_j\}_{j=0}^m$, Since for $\varphi_d = \mathcal{P}_d^*$ using equation (12) and for $\varphi_d = \mathcal{T}_d^*$ using equation (18).
- Construct the kernel matrix $\boldsymbol{\mathcal{K}}_j = \left[\mathcal{K}_{rk}^j\right]$ for each $k = \overline{0:R}$and $r = \overline{0:M}$.
- Determine the diagonal matrix $\bar{P}_i$ which are the elements $P_i(t_r), r = \overline{0:M}$.

- Compute the polynomial matrix $\mathcal{T}_t(\varphi)$ at all points $t_0, t_1, \dots, t_M$ for Legendre $\mathcal{P}_d^*$ and $\mathcal{T}_d^*$ orthogonal polynomials instead of φ.
- Taking from step 1, the positive integer number Q, and determine $w_k, (k = \overline{0{:}R})$, putting all the results in a diagonal matrix W.
- Calculate $\overline{W}$, is a row vector of weighted discrete inner product for Gauss-Legendre or Gauss-Chebyshev-Labotto definitions.

Step 3. For all $d = \overline{0{:}N}$ and each power-monomials $\ell = \overline{0{:}N}$, putting the results in step 2 into $G_d(\varphi)$ for $\varphi_d = \mathcal{P}_d^*$ and $\varphi_d = \mathcal{T}_d^*$.
Step 4. For each $d = \overline{0{:}N}$, use all results in step 3 to construct the block vector. Thus, to obtain the Tau matrix $\mathcal{H}_{\mathcal{T}}(\varphi)$.
Step 5. Construct the block vector $\mathcal{F}_{\mathcal{T}}$ by determining all elements $f(t_r), r = \overline{0{:}N}$.
Step 6. Construct the condition matrix $\mathcal{B}_\mu = [\Omega_{kd}]$ by equation (37).
Step 7. Putting all the results of steps (3, 4, 5, and 6) to complete the system (39).
Step 8. For constant coefficients $\hat{u}_d$'s $d = \overline{0{:}N}$ apply any iterative technique to system (39), which is constructed in step 7 after multiplying both sides by $\begin{bmatrix} \mathcal{H}_{\mathcal{T}}(\varphi) \\ \mathcal{B}_\mu \end{bmatrix}^T$ for SLP and SCP.
Step 9. To obtain the approximate solution $\widehat{U}_N(t)$ of $u(t)$, substitute $\hat{u}_d$'s in equation (25) for SLP $\{\mathcal{P}_d^*\}$ and in equation (26) for SCP $\{\mathcal{T}_d^*\}$.

5.2 Galerkin Method (GM)

This Method may be viewed as a modification of the Tau method. Rather than using the derivative of the residual function E_N with respect to the unknown $\hat{u}_d$'s. The derivative of the approximating function is used and defined in the approximate formula, such that they fulfill the conditions and do not have to be imposed by additional equations [11, 22]. Here, let us approximate the solution of FIDEs of Fredholm type (1) by the following finite sum:

$$u(t) \cong \widehat{U}_N(t) = \chi^*(t) + \sum_{d \le N-\mu} \hat{u}_d \chi_d(t) \tag{40}$$

where $\chi^*(t)$ satisfies the linear boundary conditions given in (2) and $\chi_d(t)$ satisfies the homogeneous boundary conditions for each $d \le N - \mu$, i.e. $\chi_d(a) = \chi_d(b) = 0$ for each $d = 0,1, \dots, N - \mu$. That is, $\widehat{U}_N(t)$ automatically satisfies the boundary conditions. There are many choices for the trial functions. $\chi_d(t)$, in this work, we chose it as a linear combination of orthogonal polynomials on a domain that satisfies the homogeneous boundary conditions. In most cases, we use basis (trial) functions of the form:

$$\chi_d(t) = \varphi_d(t) + e_{d,1}\varphi_{d+1}(t) + \cdots + e_{d,\mu}\varphi_{d+\mu}(t), \qquad d \le N - \mu \tag{41}$$

where $\varphi_d(t)$ is either the Legendre or Chebyshev polynomial of degree d, μ is the number of underlying boundary conditions, and $\{e_{d,\ell}: \ell = 0,1, \dots, \mu\}$ should be chosen so that $\chi_d(t)$ satisfies the corresponding homogeneous boundary conditions [23]. While the test functions ψ_d are chosen to be identical to trial functions, i.e,

$$\psi_d(t) = \frac{\partial U_N(t)}{\partial \hat{u}_\ell} = \chi_\ell(t), \qquad \ell \le N - \mu$$

Thus, the residual error equation (24) becomes:

$$\langle \chi_\ell, E_N(t; \hat{u}_0, \hat{u}_1, \dots, \hat{u}_N) \rangle_{M,\overline{W}} = 0, \qquad \ell = 0,1, \dots, N - \mu \tag{42}$$

substitute equations (40) into (22) and use the formulas (23) for orthogonal polynomials φ_d to obtain equation (42) as follows:

$$\sum_{d \le N-\mu} \hat{u}_d \langle \chi_\ell, L[\chi_d(t)] \rangle_{M,\overline{W}} = \langle \chi_\ell, f(t) - L[\chi^*(t)] \rangle_{M,\overline{W}}$$

Thus, after using the discrete inner product definition and L is taken as in (23), for all $\ell \le N - \mu$, we have

$$\sum_{d \le N-\mu} \hat{u}_d \left[\sum_{r=0}^{M} \overline{w}_r \chi_\ell(t_r) L[\chi_d(t_r)] \right] = \sum_{r=0}^{M} \overline{w}_r \chi_\ell(t_r) \left[f(t_r) - L[\chi^*(t_r)] \right] \tag{43}$$

where the pair $\{t_r;\bar{w}_r\}_{r=0}^{M}$ are given in the equation (32) if we apply the Legendre orthogonal or equation (33) for using the Chebyshev orthogonal polynomials, respectively. Thus, from equations (29) or (30) we get, for all $\ell \le N-\mu$:

$$\sum_{d\le N-\mu}\hat{u}_d\left\{\sum_{r=0}^{M}\bar{w}_r\chi_\ell(t_r)\left[\sum_{i=0}^{n}P_i(t_r)\,{}_a^CD_t^{\alpha_{n-i}}\chi_d(t_r)\right.\right.$$
$$\left.\left.-\frac{\lambda(b-a)}{2}\sum_{j=0}^{m}\sum_{k=0}^{R}{}'' w_k\,\mathcal{K}_j(t_r,s_k)\,{}_a^CD_s^{\beta_{m-j}}\chi_d(s_k)\right]\right\}$$
$$=\sum_{r=0}^{M}\bar{w}_r\chi_\ell(t_r)\left[f(t_r)-\sum_{i=0}^{n}P_i(t_r)\,{}_a^CD_t^{\alpha_{n-i}}\chi^*(t_r)\right.$$
$$\left.+\frac{\lambda(b-a)}{2}\sum_{j=0}^{m}\sum_{k=0}^{R}{}'' w_k\,\mathcal{K}_j(t_r,s_k)\,{}_a^CD_s^{\beta_{m-j}}\chi^*(s_k)\right] \tag{44}$$

Equation (39), in matrix form, is:

$$\mathcal{H}_G(\chi)\hat{U}^T=F_G \tag{45}$$

where $\mathcal{H}_G$ is a row-block vector, defined by

$$\mathcal{H}_G(\chi)=[G_0(\chi)\quad G_1(\chi)\quad\cdots\quad G_{N-\mu}(\chi)]$$

and

$$F_G=\Upsilon(\chi)\bar{W}F\quad;\quad F=([\bar{f}_0\quad\bar{f}_1\quad\cdots\quad\bar{f}_M]_{1\times M+1})^T$$

While, for all $d\le N-\mu$, we define

$$G_d(\chi)=\Upsilon(\chi)\bar{W}\left[\sum_{i=0}^{n}\bar{P}_i\,M_{n,i}^{\alpha}(\chi_d)-\frac{\lambda(b-a)}{2}\sum_{j=0}^{m}\boldsymbol{\mathcal{K}}_j\,WR_{m,j}^{\beta}(\chi_d)\right]$$

and

$$\bar{f}_d=f(t_d)-\sum_{i=0}^{n}P_i(t_d)\,{}_a^CD_t^{\alpha_{n-i}}\chi^*(t_d)+\frac{\lambda(b-a)}{2}\sum_{j=0}^{m}\sum_{k=0}^{R}{}'' w_k\,\mathcal{K}_j(t_d,s_k)\,{}_a^CD_s^{\beta_{m-j}}\chi^*(s_k)$$

Further,

$$\Upsilon(\chi)=\begin{bmatrix}\chi_0(t_0) & \chi_0(t_1) & \cdots & \chi_0(t_M)\\ \chi_1(t_0) & \chi_1(t_1) & \cdots & \chi_1(t_M)\\ \vdots & \vdots & \ddots & \vdots\\ \chi_{N-\mu}(t_0) & \chi_{N-\mu}(t_1) & \cdots & \chi_{N-\mu}(t_M)\end{bmatrix}_{(N-\mu+1)\times(M+1)}$$

All other parts $\bar{W},\bar{P}_i,W,M_{n,i}^{\alpha}(\chi_d),R_{m,j}^{\beta}(\chi_d)$ and $\boldsymbol{\mathcal{K}}_j$ are defined in the present section (5.1), and χ is any orthogonal linear combination that satisfies the problem.

For instance, in the actual solution procedure, it is appropriate to select the expansion functions for the Galerkin approximates to be linear combinations of shifted orthogonal (Legendre or Chebyshev) polynomials, depending on the form of the boundary conditions formula (2). Consequently, that:

(I) If $\mu=1$ that is $0<\max\{\lceil\alpha_n\rceil,\lceil\beta_m\rceil\}\le 1$, so we have only one boundary condition formed:

$$g_{11}u(a)+h_{11}u(b)=\mathcal{C}_1 \tag{46}$$

where g_{11}, h_{11} and $\mathcal{C}_1$ are given real numbers. Assume that u_a be any real number and take $u_b=(\mathcal{C}_1-g_{11}u_a)/h_{11}$ so that for choosing $\chi^*(t)$ to satisfy the non-homogeneous boundary condition, formed as:

$$\chi^*(t)=\frac{b-t}{b-a}u_a+\frac{t-a}{b-a}u_b \tag{47}$$

Further, it is transparent that the base (trial) functions $\{\chi_d(t)\}$ in equation (41) can be constructed, for $\mu=1$, by using:

i. The SCPs $\varphi_d=\mathcal{T}_d^*$ for all $d=0,1,\dots,N-1$. Thus, we set

$$\chi_d(t)=\mathcal{T}_d^*(t)+e_{d1}\mathcal{T}_{d+1}^*(t)$$

where $\mathcal{T}_k^*(t)$ is the k-th degree shifted Chebyshev polynomial (11). We will choose $\{e_{d1}\}$ such that $\chi_d(t)$ verifies the homogeneous boundary condition $g_{11}\chi_d(a) + h_{11}\chi_d(b) = 0$.
Recall equation (15), the boundary condition leads to the following equation:

$$\chi_d(a) = (-1)^d(1 - e_{d1}) \quad \text{and} \quad \chi_d(b) = 1 + e_{d1}$$

Thus, after putting it in a homogeneous boundary condition, we obtain

$$e_{d1} = \frac{g_{11}(-1)^d + h_{11}}{g_{11}(-1)^d - h_{11}}$$

The trial function to expand the solution for FIDEs of Fredholm type (1) with boundary condition (46) is formed

$$\chi_d(t) = \mathcal{T}_d^*(t) + \frac{g_{11}(-1)^d + h_{11}}{g_{11}(-1)^d - h_{11}}\mathcal{T}_{d+1}^*(t) \tag{48}$$

provided that $g_{11} \neq h_{11}$ (for d-even) and $g_{11} \neq -h_{11}$ (for d-odd). If not, we construct the trial function χ_d taking three terms of equation (41) as, for all $d = 0,1,\dots,N-2$:

$$\chi_d(t) = \mathcal{T}_d^*(t) + e_{d1}\mathcal{T}_{d+1}^*(t) + e_{d2}\mathcal{T}_{d+2}^*(t)$$

since if $\chi_d(a) = 0$ and $\chi_d(b) = 0$, we obtain the homogeneous boundary condition. Thus, equation (15) leads to the following system of $\{e_{d1}, e_{d2}\}$:

$$\left.\begin{aligned} (-1)^d + e_{d1}(-1)^{d+1} + e_{d2}(-1)^{d+2} &= 0 \\ 1 + e_{d1} + e_{d2} &= 0 \end{aligned}\right\} \tag{49}$$

Hence, $\{e_{d1}, e_{d2}\}$ can be uniquely determined from (49): $e_{d1} = 0$ and $e_{d2} = -1$ for all d. On the other hand, it is clear that $\{\chi_d(t)\}$ are linearly independent and formed as:

$$\chi_d(t) = \mathcal{T}_d^*(t) - \mathcal{T}_{d+2}^*(t), \qquad d = 0,1,\dots,N-2 \tag{50}$$

ii. The SLP's $\varphi_d = \mathcal{P}_d^*$ for all $d = 0,1,\dots,N-1$. By the same steps as in the present part (i) with using properties in equation (8) for Shifted Legendre Polynomial, we obtain the trial set:

$$\chi_d(t) = \mathcal{P}_d^*(t) + \frac{g_{11}(-1)^d + h_{11}}{g_{11}(-1)^d - h_{11}}\mathcal{P}_{d+1}^*(t) \tag{51}$$

for$d = \overline{0{:}N-1}$, provided $g_{11} \neq h_{11}$ (for d-even) and $g_{11} \neq -h_{11}$ (for d-odd). If not, also by the same procedure, we can obtain $e_{d1} = 0$ and $e_{d2} = -1$, so

$$\chi_d(t) = \mathcal{P}_d^*(t) - \mathcal{P}_{d+2}^*(t), \qquad d = \overline{0{:}N-2} \tag{52}$$

(II) If $\mu = 2$ that is $1 < \max\{\lceil\alpha_n\rceil, \lceil\beta_m\rceil\} \leq 2$, so we have two boundary conditions formed as below from equation (2)

$$[g_{11}u(a) + h_{11}u(b)] + [g_{12}u'(a) + h_{12}u'(b)] = \mathcal{C}_1$$
$$[g_{21}u(a) + h_{21}u(b)] + [g_{22}u'(a) + h_{22}u'(b)] = \mathcal{C}_2$$

Leads us to many cases, here we discuss four types:

1. All parameters $\{h_{11}, h_{12}, h_{22}, g_{12}, g_{21}, g_{22}\}$ are equal to zero, it leads to:

$$u(a) = u_a \quad (u_a \in \mathbb{R} \quad \text{and} \quad u_a = \mathcal{C}_1/g_{11})$$
$$u(b) = u_b \quad (u_b \in \mathbb{R} \quad \text{and}\ u_b = \mathcal{C}_2/h_{21})$$

For choosing $\chi^*(t)$ to satisfy these two boundary conditions, we can form:

$$\chi^*(t) = \frac{b-t}{b-a}u_a + \frac{t-a}{b-a}u_b \tag{53}$$

easily to see that $\chi^*(a) = u_a$ and $\chi^*(b) = u_b$. It remains to construct the linear combination of orthogonal polynomials for the formed $\chi_d(t)$ for all $d = 0,1,\dots,N-2$.

(i) The SCP's $\varphi_d = \mathcal{T}_d^*$. Thus, we set:

$$\chi_d(t) = \mathcal{T}_d^*(t) + e_{d1}\mathcal{T}_{d+1}^*(t) + e_{d2}\mathcal{T}_{d+2}^*(t)$$

We will choose the set $\{e_{d1}, e_{d2}\}$ such that it satisfies the two boundary conditions that are $\chi_d(a) = 0$ and $\chi_d(b) = 0$. Since equations (15) lead to the above:

$$\left.\begin{aligned} 1 - e_{d1} + e_{d2} &= 0 \\ 1 + e_{d1} + e_{d2} &= 0 \end{aligned}\right\} \tag{54}$$

After solving this linear system, we obtain the sets $\{e_{d1}, e_{d2}\}$ uniquely: $e_{d1} = 0$ and $e_{d2} = -1$ for all $d = 0,1,\dots,N-2$. Thus, the trial function is formed:

$$\chi_d(t) = \mathcal{T}_d^*(t) - \mathcal{T}_{d+2}^*(t) \tag{55}$$

This equation is equivalent, in the sense of satisfying the homogeneous boundary conditions $\chi_d(a) = \chi_d(b) = 0$, for all $d = 0,1,\dots,N-2$ to:

$$\chi_d(t) = \begin{cases} \mathcal{T}_d^*(t) - \mathcal{T}_0^*(t)\ , & d - \text{is even} \\ \mathcal{T}_d^*(t) - \mathcal{T}_1^*(t)\ , & d - \text{is odd} \end{cases} \qquad (56)$$

(ii) The SLP's $\varphi_d = \mathcal{P}_d^*$ for all $d = 0,1,\dots,N-2$. By some procedure as in part (i) with using properties in (8), we get the trial functions:

$$\chi_d(t) = \mathcal{P}_d^*(t) - \mathcal{P}_{d+2}^*(t) \qquad (57)$$

Also, equation (57) is equivalent to

$$\chi_d(t) = \begin{cases} \mathcal{P}_d^*(t) - \mathcal{P}_0^*(t)\ , & d - \text{is even} \\ \mathcal{P}_d^*(t) - \mathcal{P}_1^*(t)\ , & d - \text{is odd} \end{cases} \qquad (58)$$

Thus, equations (53) with (55 or 56) or (57 or 58) construct the approximate solution (40), which is WR-(Chebyshev) or (Legendre) Galerkin techniques for boundary conditions of type 1.

2. All parameters $\{\hbar_{11}, \hbar_{12}, \hbar_{22}, g_{11}, g_{21}, g_{22}\}$ vanish, so we have the two conditions:

$$u'(a) = u'_a \quad (u'_a \in \mathbb{R} \ \text{ and} u'_a = \mathcal{C}_1/g_{12})$$
$$u(b) = u_b \quad (u_b \in \mathbb{R} \ \text{ and } u_b = \mathcal{C}_2/\hbar_{21})$$

To satisfy these two boundary conditions, we can form $\chi^*(t)$ as:

$$\chi^*(t) = -\frac{(b-t)^2}{2(b-a)}u'_a + \left[\frac{t-a}{b-a}\right]^2 u_b \qquad (59)$$

it is easy to see that $\frac{d\chi^*(a)}{dt} = u_a$ and $\chi^*(b) = u_b$. To construct the linear combination of orthogonal polynomials $\chi_d(t); d = 0,1,\dots,N-2$, which satisfies the homogeneous boundary conditions:

(i) The SCPs $\varphi_d = \mathcal{T}_d^*$, and we set:

$$\chi_d(t) = \mathcal{T}_d^*(t) + e_{d1}\mathcal{T}_{d+1}^*(t) + e_{d2}\mathcal{T}_{d+2}^*(t)$$

since equations (15 and 16) with two homogeneous boundary conditions $\left.\frac{d\chi_d(t)}{dt}\right|_{t=a} = \chi_d(b) = 0$ leads to the following linear system:

$$\left.\begin{aligned} d^2 - e_{d1}(d+1)^2 + e_{d2}(d+2)^2 = 0 \\ 1 + e_{d1} + e_{d2} = 0 \end{aligned}\right\} \qquad (60)$$

After solving (60), we obtain the set $\{e_{d1}, e_{d2}\}$ uniquely, for all $d = 0,1,\dots,N-2$ which are:

$$e_{d1} = \frac{d^2 - (d+2)^2}{(d+1)^2 + (d+2)^2} \quad \text{and} \quad e_{d2} = -\frac{d^2 + (d+1)^2}{(d+1)^2 + (d+2)^2}$$

Thus, the orthogonal linear combination $\chi_d(t)$ which is a trial function formed:

$$\chi_d(t) = \mathcal{T}_d^*(t) + \frac{d^2 - (d+2)^2}{(d+1)^2 + (d+2)^2}\mathcal{T}_{d+1}^*(t) - \frac{d^2 + (d+1)^2}{(d+1)^2 + (d+2)^2}\mathcal{T}_{d+2}^*(t) \qquad (61)$$

(ii) The SLPs $\varphi_d = \mathcal{P}_d^*$ and we set:

$$\chi_d(t) = \mathcal{P}_d^*(t) + e_{d1}\mathcal{P}_{d+1}^*(t) + e_{d2}\mathcal{P}_{d+2}^*(t)$$

since equations (8 and 9) with boundary conditions $\chi'_d(a) = \chi_d(b) = 0$ leads to a linear system:

$$\left.\begin{aligned} d(d+1) - e_{d1}(d+1)(d+2) + e_{d2}(d+2)(d+3) = 0 \\ 1 + e_{d1} + e_{d2} = 0 \end{aligned}\right\} \qquad (62)$$

By solving system (62), we get the unique set $\{e_{d1}, e_{d2}\}$ for all $d = 0,1,\dots,N-2$, as:

$$e_{d1} = \left[\frac{d+1}{d+2}\right]^2 - 1 \quad \text{and} \quad e_{d2} = -\left[\frac{d+1}{d+2}\right]^2$$

Thus, the trial function $\chi_d(t)$ is formed as

$$\chi_d(t) = \mathcal{P}_d^*(t) + \left(\left[\frac{d+1}{d+2}\right]^2 - 1\right)\mathcal{P}_{d+1}^*(t) - \left[\frac{d+1}{d+2}\right]^2 \mathcal{P}_{d+2}^*(t) \qquad (63)$$

So, equations (59) with (61 or 63) construct the approximate solution $U_N(t)$, equation (40), which is the WR-Chebyshev or Legendre Galerkin method for boundary conditions $u'(a) = u_a,\ u(b) = u_b$.

3. All parameters $\{\hbar_{11}, \hbar_{12}, \hbar_{21}, g_{12}, g_{21}, g_{22}\}$ vanish, that it leads to

$$u(a) = u_a \quad (u_a \in \mathbb{R} \ \text{ and } u_a = \mathcal{C}_1/g_{11})$$
$$u'(b) = u'_b \quad (u'_b \in \mathbb{R} \ \text{ and } u'_b = \mathcal{C}_2/\hbar_{22})$$

To satisfy these two boundary conditions, we can form $\chi^*(t)$ as:

$$\chi^*(t) = \left[\frac{b-t}{b-a}\right]^2 u_a + \frac{(t-a)^2}{2(b-a)}u'_b \qquad (64)$$

easily to see that $\chi^*(a) = u_a$ and $\frac{d\chi^*(b)}{dt} = u'_b$. For constructing the trial function $\chi_d(t)$, $d = 0,1,\dots,N-2$, which satisfies the homogeneous boundary conditions:
(i) For SCPs $\varphi_d = \mathcal{T}_d^*$, and doing some procedure as in type (2, i) with two homogeneous boundary conditions $\chi_d(a) = \frac{d\chi_d(b)}{dt} = 0$, we have the following trial function:

$$\chi_d(t) = \mathcal{T}_d^*(t) + \frac{(d+2)^2 - d^2}{(d+1)^2 + (d+2)^2}\mathcal{T}_{d+1}^*(t) - \frac{d^2 + (d+1)^2}{(d+1)^2 + (d+2)^2}\mathcal{T}_{d+2}^*(t) \qquad (65)$$

(ii) The SLPs $\varphi_d = \mathcal{P}_d^*$. By applying the same stages as in type (2, ii), we obtain the Legendre trial function:

$$\chi_d(t) = \mathcal{P}_d^*(t) + \left(1 - \left[\frac{d+1}{d+2}\right]^2\right)\mathcal{P}_{d+1}^*(t) - \left[\frac{d+1}{d+2}\right]^2 \mathcal{P}_{d+2}^*(t) \qquad (66)$$

Thus, equations (64) with (65 or 66) construct the approximate solution $U_N(t)$, equation (40), which is WR-Chebyshev or Legendre Galerkin method for boundary conditions $u(a) = u_a$, $u'(b) = u'_b$.
4. All parameters $\{\hbar_{11}, \hbar_{12}, \hbar_{21}, g_{11}, g_{21}, g_{22}\}$ equal to zero, which leads to two boundary conditions as:

$$u'(a) = u'_a \quad (u'_a \in \mathbb{R} \text{ and } u'_a = \mathcal{C}_1/g_{12})$$
$$u'(b) = u'_b \quad (u'_b \in \mathbb{R} \text{ and } u'_b = \mathcal{C}_2/\hbar_{22})$$

To satisfy those two boundary conditions, we define $\chi^*(t)$ as form:

$$\chi^*(t) = \frac{b-a}{2}\left[\left(\frac{t-a}{b-a}\right)^2 u'_b - \left(\frac{b-t}{b-a}\right)^2 u'_a\right] \qquad (67)$$

It is easy to see that. $\frac{d\chi^*(a)}{dt} = u'_a$ and $\frac{d\chi^*(b)}{dt} = u'_b$. For constructing the trial function $\chi_d(t)$, $d = 0,1,\dots,N-2$, which satisfies the homogeneous boundary conditions:
(i) For SCPs $\varphi_d = \mathcal{T}_d^*$, from equation (41) and $\mu = 2$ we take the trial function as:

$$\chi_d(t) = \mathcal{T}_d^*(t) + e_{d1}\mathcal{T}_{d+1}^*(t) + e_{d2}\mathcal{T}_{d+2}^*(t)$$

Since taking $\ell = 1$ in equation (16) with two homogeneous boundary equations leads us to the following two linear equations:

$$\left.\begin{aligned} d^2 - e_{d1}(d+1)^2 + e_{d2}(d+2)^2 = 0 \\ d^2 + e_{d1}(d+1)^2 + e_{d2}(d+2)^2 = 0 \end{aligned}\right\} \qquad (68)$$

After solving (68), we obtain the set $\{e_{d1}, e_{d2}\}$ uniquely, for each $d = 0,1,\dots,N-2$, which conclud the trial function $\chi_d(t)$ as:

$$\chi_d(t) = \mathcal{T}_d^*(t) - \left[\frac{d}{d+2}\right]^2 \mathcal{T}_{d+2}^*(t) \qquad (69)$$

(ii) The SLPs $\varphi_d = \mathcal{P}_d^*$, from equation (41) with $\mu = 2$ we put the trial function as:

$$\chi_d(t) = \mathcal{P}_d^*(t) + e_{d1}\mathcal{P}_{d+1}^*(t) + e_{d2}\mathcal{P}_{d+2}^*(t)$$

Since equations (9) with two homogeneous boundary equations lead us to the following system:

$$\left.\begin{aligned} d(d+1) - e_{d1}(d+1)(d+2) + e_{d2}(d+2)(d+3) = 0 \\ d(d+1) + e_{d1}(d+1)(d+2) + e_{d2}(d+2)(d+3) = 0 \end{aligned}\right\} \qquad (70)$$

For uniquely $\{e_{d1}, e_{d2}\}$ solving the system (70) to obtain the trial function formed

$$\chi_d(t) = \mathcal{P}_d^*(t) - \frac{d}{d+2}\left(\frac{d+1}{d+3}\right)\mathcal{P}_{d+2}^*(t) \qquad (71)$$

Hence, equations (67) with (69 or 71) construct the approximate solution $U_N(t)$, equation (40), which is the WR-Chebyshev or Legendre Galerkin method for boundary conditions $u'(a) = u'_a$, $u'(b) = u'_b$.
The Algorithm (AGM). The approximate solution of multi-higher IFDEs of Fredholm type (1) and boundary conditions (2) by applying for $\mu = 1$ and 2 the discrete WR-Tau method with orthogonal polynomials SLPs and SCPs can be summarized in the following stages:
Step 1.

- Input N-number of approximate terms, (R, Q)-number of terms in the quadrature integration formula and M-number of points in the discrete inner product, which are the Gauss-Legendre or Gauss-Chebyshev-Labotto collocation points.

- Set $t_r=\frac{b-a}{2}\tau_r+\frac{b+a}{2}$ where τ_r 's are the $M+1$ roots of $\mathcal{P}_{M+1}(t)$ with $\bar{w}_r=2/\{(1-\tau_r^2)[P'_{M+1}(\tau_r)]^2\}$ or $\tau_r=-\cos(r\pi/M)$ with $\bar{w}_r=\pi/2M$ for $r=0,M$ and otherwise $\bar{w}_r=\pi/M$.
- Put $s_k=\frac{b-a}{2}\eta_k+\frac{b+a}{2}$ where $\eta_k=\cos\left(\frac{k\pi}{R}\right)$, for each $k=\overline{0{:}R}$ and given Q , with

$$w_k=\frac{4}{R}\sum_{\substack{\ell=0\\ even}}^{Q}{}'' \frac{1}{1-\ell^2}cos\left(k\ell\frac{\pi}{Q}\right)$$

Step 2. For all $d=0,1,\dots,N-\mu$:

- Evaluate $M_{n,i}^{\alpha}(\chi_d)$ and $R_{m,j}^{\beta}(\chi_d)$ for all fractional orders $\{\alpha_i\}_{i=0}^{n}$ and $\{\beta_j\}_{j=0}^{m}$.
- Construct the kernel matrix $\boldsymbol{\mathcal{K}}_j=\left[\mathcal{K}_{rk}^{j}\right]$ for each $k=\overline{0{:}R}$and $r=\overline{0{:}M}$.
- Determine the diagonal matrix $\bar{P}_i$ which are the elements $P_i(t_r), r=\overline{0{:}M}$.
- Compute the matrix $\mathrm{Y}(\chi)$, all elements $\chi_d(t_\ell)$ for $\ell=\overline{0{:}M}$.
- Taking from step 1 the positive integer number Q and determine $w_k,(k=\overline{0{:}R})$, putting all the results in a diagonal matrix W.
- Calculate $\overline{W}$, is a row vector of weighted discrete inner product for Gauss-Legendre or Gauss-Chebyshev-Labotto definitions.

Step 3. For all $d=\overline{0{:}N-\mu}$ and each linear combinations χ_d putting the results in step 2 into $G_d(\chi)$ for Legendre and Chebyshev orthogonals.

Step 4. Construct the Galerkin matrix $\mathcal{H}_G$ at each linear combination χ_d, using all results in steps 2 and 3.

Step 5. Construct the vector $\mathcal{F}_G$ by determining each element $\bar{f}_d$, $d=\overline{0{:}N-\mu}$.

Step 6. Putting all results in steps (3, 4, and 5) to complete the system (45).

Step 7. For constant coefficients $\hat{u}_d$'s $d=\overline{0{:}N}$ apply any iterative technique to system (45), which is constructed in step 7.

Step 8. To obtain the approximate solution $\widehat{U}_N(t)$ of $u(t)$, substitute $\hat{u}_d$'s in equation (25) for SLP $\{\mathcal{P}_d^*\}$ and in equation (26) for SCP $\{\mathcal{T}_d^*\}$

6. Numerical Experiments

We used the current algorithms (ATM and AGM) in this section. A MATLAB (V.9.7) application was used to run these algorithms on the computer. The values of $\sum_{i=0}^{\mathcal{M}}[u(t_i)-u_N(t_i)]^2, \mathcal{M}\in\mathbb{N}$ at $\mathcal{M}$-selected points t_i for each case in the tables are the least square errors. The running time is also given in tabular form. Additionally, apply the absolute error function $E(x_\ell)=|u(x_\ell)-u_N(x_\ell)|$, for all $\ell=0,1,\dots,\mathcal{M}$, to each test problem at the chosen interval points. Additionally, use graphs to show it.

Test problem 1. Consider a higher-order linear IFDE of Fredholm type with variable coefficients for a fractional order that lies in $(0,1)$ on the closed bounded interval $[a,b]=[0,1]$:

$${}_0^CD_t^{0.7}u(t)+\sinh(t)u(t)=\frac{6}{\Gamma(2.3)}t^{1.3}+\sinh(t)(3t^2+2)-\frac{6e^t}{4.2\Gamma(2.2)}-\frac{6}{3.5\Gamma(2.5)}t^2+\frac{6}{\Gamma(3.5)}-5e^{t+1}+8e^t+\int_0^1\left[s^2e^t\,{}_0^CD_s^{0.8}u(s)+(st^2-1)\,{}_0^CD_s^{0.5}u(s)+e^{s+t}u(s)\right]ds$$

subjected to the boundary conditions: $u(0)+u(1)=7$, while the exact solution is: $u(t)=3t^2+2$.

Here, we have: $n=1, m=2$, $\alpha_1=0.7, \alpha_0=0$ and $\beta_2=0.8, \beta_1=0.5, \beta_0=0$. So, $\mu=\max\{\lceil 0.7\rceil,\lceil 0.8\rceil\}=1$. Also, $P_1(t)=\sinh(t)$, $\mathcal{K}_0(t,s)=s^2e^t$, $\mathcal{K}_1(t,s)=st^2-1$, $\mathcal{K}_2(t,s)=e^{s+t}$ and $\lambda=1$, while $f(t)=\frac{6}{\Gamma(2.3)}t^{1.3}+\sinh(t)(3t^2+2)-\frac{6e^t}{4.2\Gamma(2.2)}-\frac{6}{3.5\Gamma(2.5)}t^2+\frac{6}{\Gamma(3.5)}-5e^{t+1}+8e^t$. Assuming $N=2$, we are going to evaluate the following approaches, establishing the approximate solution $U_N(t)$ to $u(t)$:

$$\widehat{U}_2(t)=\begin{cases}\sum_{d=0}^{2}\hat{u}_d^L\mathcal{P}_d(2t-1) & \text{for apply SLPs}\\ \sum_{d=0}^{2}\hat{u}_d^C\mathcal{T}_d(2t-1) & \text{for apply SCPs}\end{cases}\tag{72}$$

Apply the algorithms [TM and GM] to find the parameters $\hat{u}_d = \{\hat{u}_d^L, \hat{u}_d^C: d = 0,1,2\}$ in the approximate solution applying SLPs and SCPs, respectively. M-number of points in the discrete inner product, which are the Gauss-Legendre or Gauss-Chebyshev-Labotto collocation points, equals 4 in this case, while R is the number of terms in the quadrature integration formula (20) equals 10. Programs created especially for this purpose are run after the values (N, M, R) equal $(2,4,10)$ for our test problem are entered. Following the completion of each algorithmic step, the following results are obtained using the final matrix structures for the techniques and Galerkin methods-which can be solved using any iterative numerical method to determine the parameters $\{\hat{u}_d: d = 0,1,2\}$ for the SLPs and SCPs formed in Table 1:

Table 1. Presents the value of Tau-coefficients $\{\hat{u}_d\}$ using Legendre and Chebyshev OPs, respectively.

Methods / Coefficients		Discrete WR-Tau method $N = 2, M = 4, R = 10$	Discrete WR-Tau method $N = 2, M = 7, R = 25$
SLPs	$\hat{u}_0^L$	2.99999997965	2.99999999985
	$\hat{u}_1^L$	1.50000084543	1.50000000955
	$\hat{u}_2^L$	0.500000020564	0.500000000147
SCPs	$\hat{u}_0^C$	3.12499996164	3.12499999985
	$\hat{u}_1^C$	1.50000083792	1.50000000977
	$\hat{u}_2^C$	0.375000001518	0.374999999995

For SLPs and SCPs, respectively, the approximation expression forms are obtained by inserting Tau-coefficients $\{\hat{u}_d\}$ in equation (72). These results, for $(N, M, R) = (2,4,10)$:

$$\hat{U}_2^{TM-L}(t) = 3.000000123381220t^2 + 0.000001567488191t + 1.999999154779139$$

$$\hat{U}_2^{TM-C}(t) = 3.0000000121438317t^2 + 0.0000016637034979t + 1.9999991252353$$

and for $(N, M, R) = (2,7,25)$:

$$\hat{U}_2^{TM-L}(t) = 3.0000000008845772t^2 + 0.000000018222997t + 1.99999999044858$$

$$\hat{U}_2^{TM-C}(t) = 2.999999999960174t^2 + 0.000000019582633t + 1.999999990078551$$

In the Galerkin WR-method, the approximate solution depends on given boundary conditions which are selected from the fractional orders, here $\mu = 1$, and we have one boundary condition, which is $u(0) + u(1) = 7$, since $g_{11} = 1$, $\hbar_{11} = 1$ and $\mathcal{C}_1 = 7$. From equation (40), we put the approximation on all $d \leq N - 2 (= 0, \text{here for } N = 2)$, form as:

$$\hat{U}_2(t) = \chi^*(t) + \sum_{d=0}^{0} \hat{u}_d \chi_d(t) \tag{73}$$

That is, we have one Galerkin coefficient unknown $\{\hat{u}_d: d \leq N - 2\}$. Assume that $u_a = 2$ (any arbitrary real number), so we can calculate $u_b = (\mathcal{C}_1 - g_{11} u_a)/\hbar_{11} = 5$, from equation (47) we obtain $\chi^*(t) = 2(1 - t) + 5t$, which satisfies the non-homogeneous boundary condition. Also, from equations (50) and (52) we formed $\chi_d(t)$ using Chebyshev and Legendre orthogonal polynomials, respectively:

$$\chi_d(t) = \mathcal{T}_d(2t - 1) - \mathcal{T}_{d+2}(2t - 1) \quad \text{and} \quad \chi_d(t) = \mathcal{P}_d(2t - 1) - \mathcal{P}_{d+2}(2t - 1)$$

This satisfies the homogeneous boundary condition.

Using the program designed for this purpose in MATLAB, determine the values of $\hat{u}_d$'s using SLP and SCP, respectively, and uses the algorithm AGM to figure out the Legendre and Chebyshev parameters in the approximate solution from $\hat{U}_2(t)$. Consequently, the Legendre parameter $\hat{u}_0^L$ is equal to (0.499997585625) for SLPs and Chebyshev-parameter $\hat{u}_0^C$ is equal to (0.374997558336)for applying SCPs. The approximation expression for the approximate formula becomes:

$$\hat{U}_2^{GM-L}(t) = 2.99998551374797t^2 + 0.000014486252029t + 2.0$$

$$\hat{U}_2^{GM-C}(t) = 2.999980466690659t^2 + 0.0000195333093411t + 2.0$$

For discrete WR-(Tau and Galerkin) Methods using Chebyshev and Legendre polynomials, Table (2) compares the approximate solution $\widehat{U}_2(t)$ with the exact solution $u(t)$. The least square error (LSE), running time, and residual error $E_2\left(t;\widehat{U}_2\right) = LSE_u$, applying (27 and 28), which are independent of the given solutions, are all included in this table.

Table 2. Numerical results for different values of t and comparison between methods at $N = 2$, $M = 4$ and $R = 10$, for problem 1.

t_r	Exact Solution	Proposed Method: Discrete WRM-SLPs		Proposed Method: Discrete WRM-SCPs		Quadrature Method
		TM	GM	TM	GM	Midpoint Type ([21])
0.0	2.00	1.9999992	2.00	1.9999991	2.00	1.9779087104
0.1	2.03	2.0299993	2.0300013	2.0299993	2.0300018	2.0203666961
0.2	2.12	2.1199995	2.1200023	2.1199995	2.1200031	2.1201273765
0.3	2.27	2.2699996	2.270003	2.2699996	2.2700041	2.2780889009
0.4	2.48	2.4799998	2.4800035	2.4799998	2.4800047	2.4945824415
0.5	2.75	2.75	2.7500036	2.75	2.7500049	2.7697732560
0.6	3.08	3.0800001	3.0800035	3.0800001	3.0800047	3.1037712876
0.7	3.47	3.4700003	3.470003	3.4700003	3.4700041	3.4966686664
0.8	3.92	3.9200005	3.9200023	3.9200005	3.9200031	3.9485538829
0.9	4.43	4.4300007	4.4300013	4.4300006	4.4300018	4.4595144645
1.0	5.00	5.0000008	5.00	5.0000008	5.00	5.0296387137
$LSE.$		3.149931×10^{-12}	$6.9943504 \times 10^{-11}$	3.105766×10^{-12}	1.27171×10^{-10}	5.091074×10^{-03}
$E_2 = LSE_u.$		1.576186×10^{-11}	4.601864×10^{-10}	1.77778×10^{-11}	8.36707×10^{-10}	
$R.Time /sec$		2.29851	1.36220	1.41287	1.19934	0.872449

Table 3. Comparison of fixed $N = 2$ and various inputs of approximation integral parts R and discrete inner product terms M based on residual error functions, least square error, and running time

The Methods		$(M, R) = (4, 25)$			$(M, R) = (7, 100)$		
		$LSE.$	$LSE_u.$	$R.Time$	$LSE.$	$LSE_u.$	$R.Time$
SLPs	TM	4.010198×10^{-16}	2.00177×10^{-15}	1.855945	3.716498×10^{-22}	3.716498×10^{-21}	3.370705
	GM	8.932091×10^{-15}	5.876781×10^{-15}	1.42733	8.232923×10^{-21}	5.416802×10^{-20}	2.949015
SCPs	TM	3.950149×10^{-16}	2.267602×10^{-15}	1.442218	3.888557×10^{-22}	2.035401×10^{-21}	3.225638
	GM	1.620367×10^{-14}	1.066104×10^{-13}	1.272886	1.59368×10^{-20}	1.04855×10^{-19}	2.88771

For test problem 1, Figure 1 compares the exact answer with a step size of $h = 0.1$ and $N = 2$ with the approximate solutions obtained by algorithms (ATM and AGM) for SLPs and SCPs. Figure 2 shows the absolute errors as well.

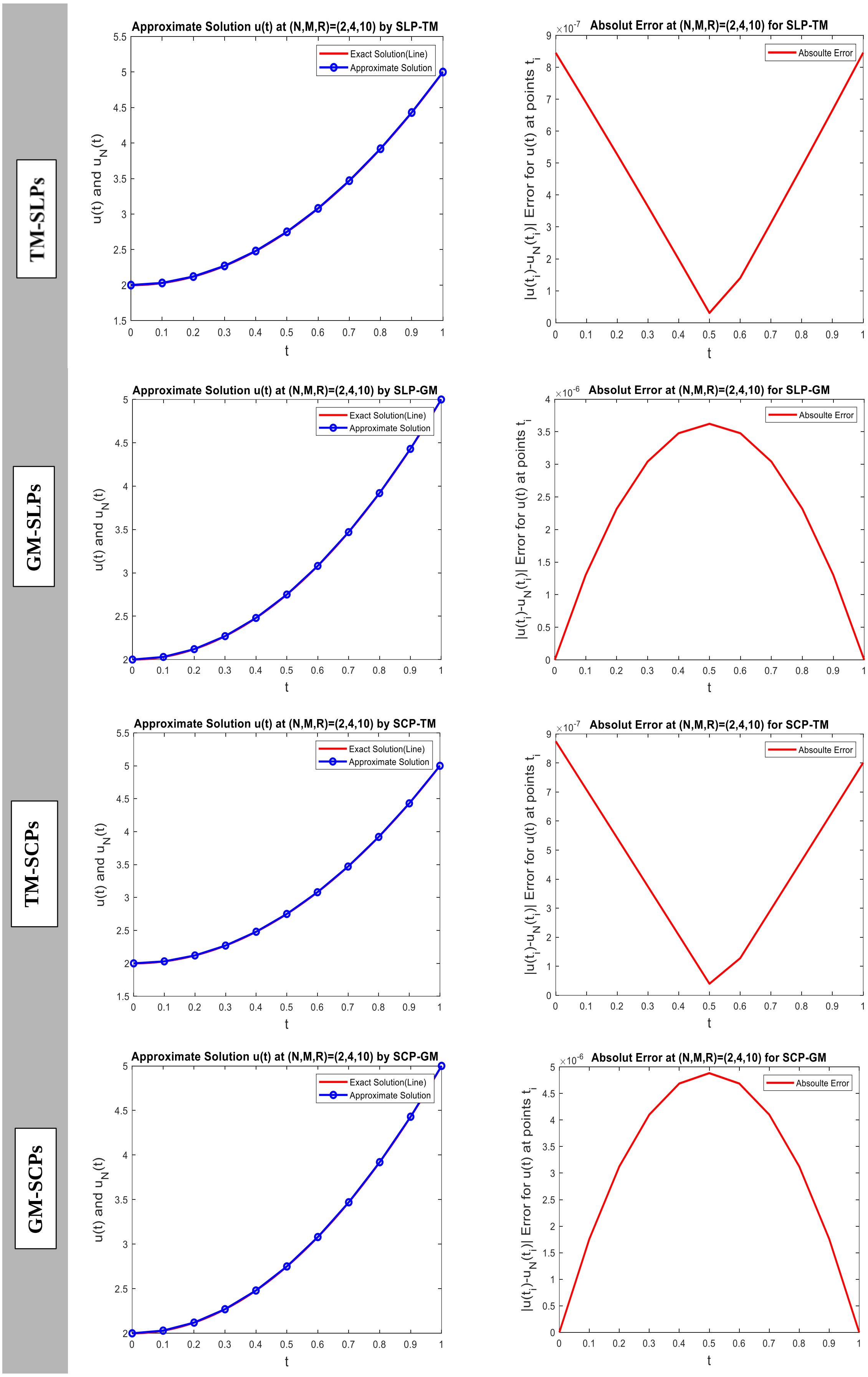
TM-SLPs
Approximate Solution u(t) at (N,M,R)=(2,4,10) by SLP-TM
Exact Solution(Line)
Approximate Solution
u(t) and u_N(t)
t
Absolut Error at (N,M,R)=(2,4,10) for SLP-TM
Absoulte Error
|u(t_i)-u_N(t_i)| Error for u(t) at points t_i
GM-SLPs
Approximate Solution u(t) at (N,M,R)=(2,4,10) by SLP-GM
Absolut Error at (N,M,R)=(2,4,10) for SLP-GM
TM-SCPs
Approximate Solution u(t) at (N,M,R)=(2,4,10) by SCP-TM
Absolut Error at (N,M,R)=(2,4,10) for SCP-TM
GM-SCPs
Approximate Solution u(t) at (N,M,R)=(2,4,10) by SCP-GM
Absolut Error at (N,M,R)=(2,4,10) for SCP-GM

Fig.1. For test problem 1 with a step size of $h = 0.1$, the approximate solutions, while the precise solution is shown by a solid line.

Fig.2. Absolute error plot function $|u(t) - u_N(t)|$ for $N = 2$, $h = 0.1$ for test problem 1.

Test problem 2. Consider a higher-order linear IFDE of Fredholm type with variable coefficients for fractional orders α and β on the closed bounded interval $[a, b] = [0,1]$:

$$ {}_0^CD_t^{2\alpha}u(t) - t^2\, {}_0^CD_t^{\alpha}u(t) + \sin(t)u(t) = f(t) + \int_0^1 \left[2st^2\, {}_0^CD_s^{\beta}u(s) + (1 + st^2)u(s)\right] ds $$

where

$$ f(t) = \frac{12}{\Gamma(4-2\alpha)}t^{3-2\alpha} - \frac{6}{\Gamma(3-2\alpha)}t^{2-2\alpha} - \frac{12}{\Gamma(4-\alpha)}t^{5-\alpha} + \frac{6}{\Gamma(3-\alpha)}t^{4-\alpha} + (2t^3 - 3t^2 + 1)\sin(t) - \left[\frac{24}{(5-\beta)\Gamma(4-\beta)} - \frac{12}{(4-\beta)\Gamma(3-\beta)} + \frac{3}{20}\right]t^2 - \frac{1}{2} $$

Subjected to the boundary conditions:
if $0 < \alpha \le 0.5$ and $0 < \beta \le 1$ then the boundary condition is: $u(0) + u(1) = 1$.
if $0.5 < \alpha \le 1$ and $0 < \beta \le 1$ then the boundary conditions are formed:

$$ B_1:\begin{cases} u(0) = 1 \\ u(1) = 0 \end{cases} \quad B_2:\begin{cases} u'(0) = 0 \\ u(1) = 0 \end{cases} \quad B_3:\begin{cases} u(0) = 1 \\ u'(1) = 0 \end{cases} \quad B_4:\begin{cases} u'(0) = 0 \\ u'(1) = 0 \end{cases} $$

While the exact solution is: $u(t) = 2t^3 - 3t^2 + 1$.
Here, from the state of test problem 2, we have: $n = 2, m = 1, P_1(t) = -t^2, P_2(t) = \sin(t)$, and $\mathcal{K}_0(t,s) = 2st^2$, $\mathcal{K}_1(t,s) = 1 + st^2$ with $\lambda = 1$. Now, assuming that $N = 3$, we are going to evaluate the following approaches, the approximate solution $\widehat{U}_N(t)$ to $u(t)$:

$$ \widehat{U}_3(t) = \begin{cases} \sum_{d=0}^{3} \hat{u}_d^L \mathcal{P}_d(2t-1) & \text{for apply SLPs} \\ \sum_{d=0}^{3} \hat{u}_d^C \mathcal{T}_d(2t-1) & \text{for apply SCPs} \end{cases} \tag{74} $$

To determine the parameters $\hat{u}_d = \{\hat{u}_d^L, \hat{u}_d^C : d = 0,1,2,3\}$ in the approximate solution (74) using SLPs and SCPs, respectively, use the algorithms [TM and GM]. After the values (N, M, R) equal (3,5,10) for our test problem are entered, programs designed specifically for this purpose are executed. **First**, once each algorithmic step has been completed, the parameters $\{\hat{u}_d : d = 0,1,2,3\}$ for the WR-Tau method of the SLPs and SCPs produced in the following Table 4, which are determined for fractional orders $(\alpha, \beta) = (0.3, 0.7)$ with a boundary condition $u(0) + u(1) = 1$, that is $g_{11} = 1$, $\hbar_{11} = 1$ and $\mathcal{C}_1 = 1$.

Table 4. Presents the value of Tau-coefficients $\{\hat{u}_d : d = \overline{0:3}\}$ for (N, M, R)=(3,5,10) using Legendre and Chebyshev OPs, respectively.

Methods in (OPs)	**Coefficients in the discrete WR-Tau method for $\alpha = 0.3$ and $\beta = 0.7$**			
SLPs	$\hat{u}_0^L$	$\hat{u}_1^L$	$\hat{u}_2^L$	$\hat{u}_3^L$
	0.500000160813	−0.600000229037	−0.0000001606789	0.09999998583
SCPs	$\hat{u}_0^C$	$\hat{u}_1^C$	$\hat{u}_2^C$	$\hat{u}_3^C$
	0.50000010822	−0.56250019543	−0.0000001109951	0.0624999876207

For SLPs and SCPs, respectively, the approximation expression forms are obtained by inserting Tau-coefficients $\{\hat{u}_d : d = 0,1,2,3\}$ in equation (74). This leads to:

$$ \widehat{U}_2^{TM-L}(t) = 1.99999971659933437t^3 - 3.00000053897286755t^2 + 0.00000033595900146t + 1.00000024334131066 $$

and

$$ \widehat{U}_2^{TM-C}(t) = 1.99999960386329015t^3 - 3.00000293756039135t^2 + 0.000000274274718034t + 1.00000020503410175 $$

In the WR-Galerkin method, the approximate solution depends on given boundary conditions, which are selected from the fractional orders: For fractional order $\alpha = 0.3$ and $\beta = 0.7$ that is $\mu = 1$, so we have only one boundary condition, which is $u(0) + u(1) = 1$, since $\mathscr{g}_{11} = 1$, $\hbar_{11} = 1$ and $\mathcal{C}_1 = 1$. From equation (40), we put the approximate form as:

$$\widehat{U}_3(t) = \chi^*(t) + \sum_{d=0}^{1} \hat{u}_d \, \chi_d(t) \tag{75}$$

Assume that $u_a = 1$ (any arbitrary real number) so $u_b = (\mathcal{C}_1 - \mathscr{g}_{11} u_a)/\hbar_{11} = 0$, from equation (47) we obtain $\chi^*(t) = t + 1$, which satisfies the non-homogeneous boundary condition. Also, from equations (50) and (52), we formed $\chi_d(t)$ using Chebyshev and Legendre orthogonal polynomials, respectively:

$$\chi_d(t) = \mathcal{T}_d(2t-1) - \mathcal{T}_{d+2}(2t-1) \quad \text{and} \quad \chi_d(t) = \mathcal{P}_d(2t-1) - \mathcal{P}_{d+2}(2t-1)$$

Which satisfy the homogeneous boundary condition. Apply the algorithm AGM to find the parameters $\hat{u}_d$'s in the approximate solution from $U_3(t)$, thus run the MATLAB program to obtain the following results. Consequently, for $(N, M, R) = (3,5,10)$ the Legendre-parameter: $\hat{u}_0^L = 0.000000045085$, $\hat{u}_1^L = 0.100000136695$ for SLPs and Chebyshev-parameter $\hat{u}_0^C = 0.00000008073$, $\hat{u}_1^C = 0.0625001544305$ for apply SCPs. The approximation expression for the approximate formula (75), becomes:

$$\widehat{U}_3^{GM-L}(t) = 2.000002733891921t^3 - 3.0000038303288789t^2 + 0.0000010964369579t + 1.0$$

$$\widehat{U}_3^{GM-C}(t) = 2.00000494177656796t^3 - 3.0000067667999537t^2 + 0.0000018250233857t + 1.0$$

For discrete WR-(Tau and Galerkin) Methods using Chebyshev and Legendre polynomials, Table (5) compares the approximate solution $U_3(t)$ with the exact solution $u(t)$. The least square error (LSE), running time, and residual error $E_3\big(t; \widehat{U}\big)$, applying (27 and 28), which are independent of the given solutions, are all included in this table.

Table 5. Numerical results for different values of t and comparison between methods at $N = 3$, $M = 5$ and $R = 10$, for $\alpha = 0.3$ and $\beta = 0.7$ in test problem 2.

t_r	**Exact Solution**	**Proposed Method: Discrete WRM-SLPs**		**Proposed Method: Discrete WRM-SCPs**		**Quadrature Method Midpoint Type ([21])**
		TM	**GM**	**TM**	**GM**	
0.0	1.0	1.0000002	1.0	1.0000002	1.0	1.0118444092
0.1	0.972	0.97200027	0.97200007	0.97200023	0.97200012	0.9764714336
0.2	0.896	0.89600029	0.89600009	0.89600024	0.89600013	0.8974204538
0.3	0.784	0.78400029	0.78400006	0.78400025	0.78400007	0.7839696749
0.4	0.648	0.64800027	0.648	0.64800024	0.64799996	0.6472104466
0.5	0.500	0.50000024	0.49999993	0.50000022	0.49999984	0.4986831141
0.6	0.352	0.35200019	0.35199987	0.35200018	0.35199973	0.3500782391
0.7	0.216	0.21600012	0.21599983	0.21600012	0.21599966	0.2131234868
0.8	0.104	0.10400002	0.10399983	0.10400003	0.10399966	0.0995072617
0.9	0.028	0.027999903	0.02799987	0.027999925	0.027999764	0.0207673005
1.0	0.000	−0.00000024	0.000	−0.00000021	0.000	−0.011703169
$LSE.$		5.495861×10^{-13}	1.131823×10^{-13}	4.205493×10^{-13}	4.291566×10^{-13}	3.860882×10^{-04}
$E_2 = LSE_u.$		$2.0755145 \times 10^{-13}$	7.31802×10^{-13}	1.829171×10^{-13}	2.564156×10^{-13}	
$R.Time /sec$		1.76679	1.845444	1.632162	1.65464	1.242617

Table 6. Comparison of various inputs: approximation terms N of approximation integral parts R and discrete inner product terms M based on residual error functions, least square error, and running time, for $\alpha = 0.3$ and $\beta = 0.7$ in test problem 2.

The Methods		$(N, M, R) = (3, 4, 25)$			$(N, M, R) = (5, 8, 100)$		
		$LSE.$	$LSE_u.$	$R.Time$	$LSE.$	$LSE_u.$	$R.Time$
SLPs	TM	3.027579×10^{-18}	1.143035×10^{-18}	1.656561	3.251865×10^{-26}	1.322686×10^{-26}	6.551402
	GM	$6.1979825 \times 10^{-19}$	4.015097×10^{-18}	1.634926	2.854353×10^{-26}	4.068574×10^{-25}	6.728294
SCPs	TM	2.139374×10^{-18}	9.33096×10^{-19}	1.483473	$2.8283001 \times 10^{-26}$	1.206039×10^{-26}	6.177873
	GM	2.515776×10^{-18}	1.48891×10^{-17}	1.57148	1.712755×10^{-25}	2.97643×10^{-24}	6.49696

Figure.3. displays the absolute errors for test problem 2 with a step size of h=0.1 and N=3, where the approximate solutions for SLPs and SCPs were obtained using algorithms (ATM and AGM).

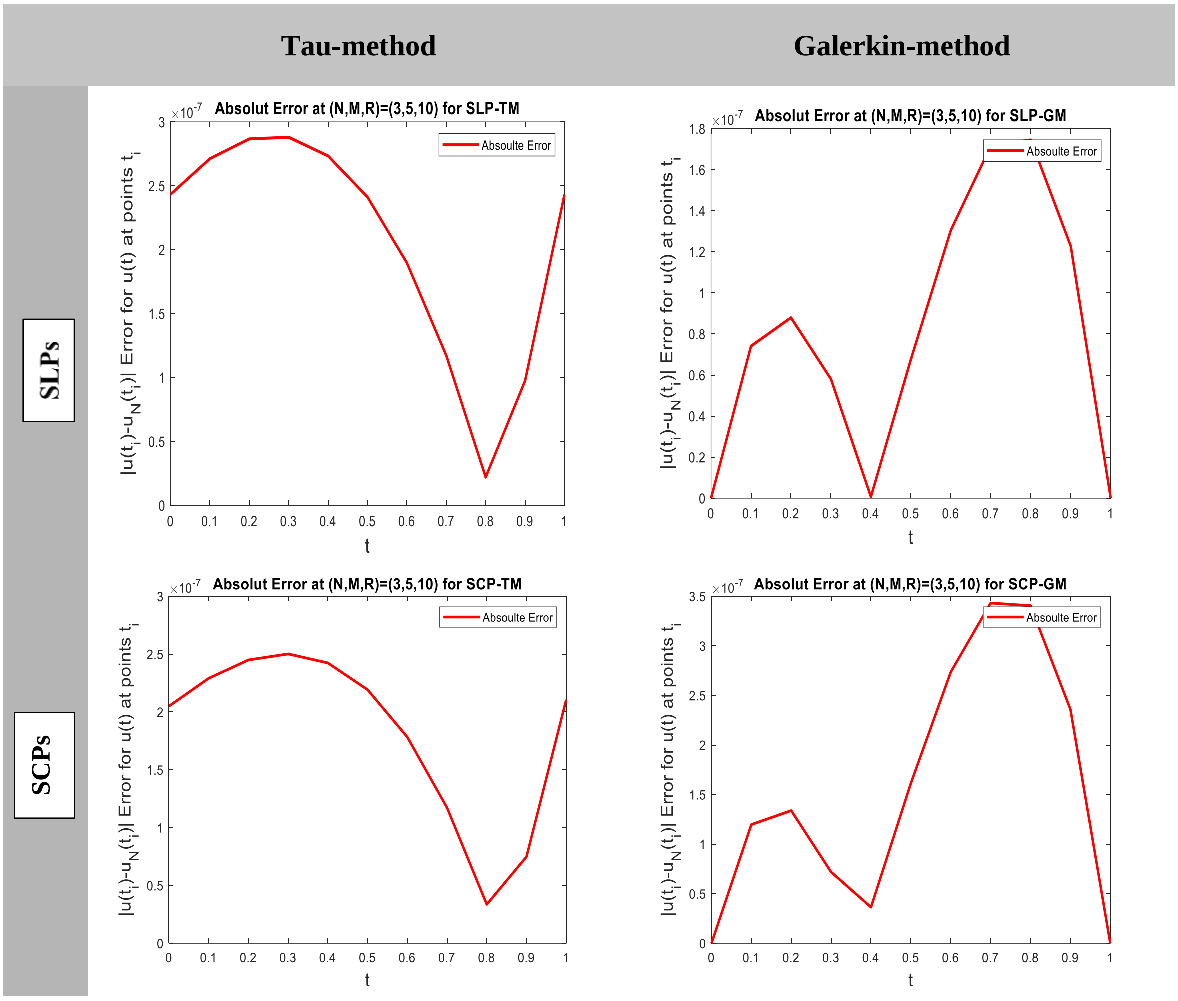

Fig. 3. Absolute error plot function $|u(t) - u_N(t)|$ for $N = 3, h = 0.1$ and for fractional orders $\alpha = 0.3$ and $\beta = 0.7$ in test problem 2.

Second, taking into account the boundary condition $\{B_1 \; or \; B_2 \; or \; B_3 \; or \; B_4\}$and the fractional orders $(\alpha, \beta) = (0.7,0.7)$ We were able to get around this problem by applying the techniques described in Section 5 as well as the ATM and AGM algorithms. The parameters $\{\hat{u}_d\}$ for the SLPs and SCPs, respectively, that are produced, and the approximate solutions are obtained from equations (74 and

75). When using equations (27 and 28), respectively, the errors for $(N, M, R) = (3,4,10)$ and $(5,8,100)$ are based on running time, least square error, and residual equations $E_3\left(t; \widehat{U}_3\right)$, which do not rely on existing solutions. These errors are displayed in Tables 8 and 9. It shows that by employing a few terms of SLPs and SCPs, we may obtain a good approximation of the exact solution.

- The Tau-approximation formulas for each boundary condition and fractional orders $(\alpha, \beta) = (0.7,0.7)$ for inputs $(N, M, R) = (3,4,10)$. To construct the approximate expression (74) for each boundary condition:

For SLPs:

$$\widehat{U}_3^{TM-L_{B_1}}(t) = 1.999999855682927t^3 - 2.999999955264605t^2 + 0.0000000067444832t + 1.0000000072934977$$

$$\widehat{U}_3^{TM-L_{B_2}}(t) = 1.999999854806985t^3 - 2.999999901420348t^2 + 0.00000000026343135t + 1.000000053321494$$

$$\widehat{U}_3^{TM-L_{B_3}}(t) = 1.999999925070186t^3 - 2.99999992735773t^2 + 0.000000079153717t + 0.9999999986796200$$

$$\widehat{U}_3^{TM-L_{B_4}}(t) = 1.999999873119297t^3 - 2.9999998173363459t^2 + 0.000000011116315t + 1.0000001102265565$$

and for SCPs:

$$\widehat{U}_3^{TM-C_{B_1}}(t) = 1.999999860569862t^3 - 2.999999923778280t^2 + 0.0000000214248328t + 1.0000000149832052$$

$$\widehat{U}_3^{TM-C_{B_2}}(t) = 1.999999848985222t^3 - 2.99999991111015334t^2 + 0.0000000056120262t + 1.000000034096616$$

$$\widehat{U}_3^{TM-C_{B_3}}(t) = 1.9999999308351404t^3 - 2.999999940927739t^2 + 0.000000086679368t + 0.999999983548512$$

$$\widehat{U}_3^{TM-C_{B_4}}(t) = 1.999999908563937t^3 - 2.999999899637236t^2 + 0.000000056251807t + 1.0000000179574973$$

- The Galerkin-approximation formulas for each boundary condition and fractional orders $(\alpha, \beta) = (0.7,0.7)$ for inputs $(N, M, R) = (3,4,10)$. To construct the approximate expression (40) for each boundary condition:

For B_1: here, $u_a = 1, u_b = 0$ and the equation (53) becomes $\chi^*(t) = 1 - t$. Further, using equation (55) or (56) for SCP and equation (57) or (58) for SLP to form $\chi_d(t)$.

For B_2:here $u_a = 1, u_b' = 0$ and equation (64) becomes$\chi^*(t) = -(1-t)^2$. Further, using equation (65) for SCP and (66) for SLP, respectively. To define $\chi_d(t)$ that is, it satisfies the homogeneous boundary conditions.

For B_3: here $u_a' = 0, u_b = 0$ and equation (59) become a zero function. Further, using equation (61) for SCP and (63) for SLP to define $\chi_d(t)$ that is, it satisfies the homogeneous boundary conditions.

For B_4:here $u_a' = 0, u_b' = 0$ and equation (67) becomes a zero function. Further, using equation (69) for SCP and (71) for SLP to define $\chi_d(t)$ that is, it satisfies the homogeneous boundary conditions.

Apply the algorithm AGM for each type of boundary conditions to obtain the values of the parameters $\hat{u}_d$'s. Thus, the following approximate formulas are obtained:

For SLPs:

$$\widehat{U}_3^{GM-L_{B_1}} = 1.9999999049473319t^3 - 2.999999952740269t^2 + 0.000000047792937t + 1.0$$

$$\widehat{U}_3^{GM-L_{B_2}} = 1.9999998611580086t^3 - 2.9999999115401067t^2 - 7.3468397 \times 10^{-40}t + 1.0000000503820981$$

$$\widehat{U}_3^{GM-L_{B_3}} = 1.9999999274281567t^3 - 2.99999993111954275t^2 + 0.000000079954615t + 1.0$$

$\widehat{U}_3^{GM-L_{B_4}} = 1.9999998459663774t^3 - 2.9999997689495661t^2 + 1.00000013135834713$

and for SCPs:

$$\widehat{U}_3^{GM-C_{B_1}}(t) = 1.9999999030601252t^3 - 2.99999995153816576t^2 + 0.000000048478040\,55t + 1.0$$

$$\widehat{U}_3^{GM-C_{B_2}}(t) = 1.9999998432053852t^3 - 2.9999998877573466t^2 + 3.673419846 \times 10^{-40}t + 1.0000000445519613885$$

$$\widehat{U}_3^{GM-C_{B_3}}(t) = 1.99999992566183573t^3 - 2.99999992829598662t^2 + 0.0000000796064661t + 1.0$$

$$\widehat{U}_3^{GM-C_{B_4}}(t) = 1.9999998335236502\,57t^3 - 2.99999975028547539t^2 + 1.0000000890230156999$$

Table 8. In Proposed Method: Discrete WRM-SLPs, comparison of N-limited terms and various inputs of approximation integral parts R and discrete inner product terms M based on residual error functions, least square error, and running time.

The Methods		$(N,M,R)=(3,4,10)$			$(N,M,R)=(5,8,100)$		
		$LSE.$	$LSE_u.$	$R.Time$	$LSE.$	$LSE_u.$	$R.Time$
SLPs	TM_{B_1}	1.210103×10^{-14}	1.887045×10^{-13}	1.630737	1.65484×10^{-27}	1.17776×10^{-26}	6.492612
	TM_{B_2}	2.814049×10^{-14}	1.971609×10^{-13}	1.55249	5.23644×10^{-28}	1.18466×10^{-26}	6.460190
	TM_{B_3}	2.80244×10^{-14}	2.052618×10^{-13}	1.518952	1.71677×10^{-27}	1.21794×10^{-26}	6.507928
	TM_{B_4}	2.371803×10^{-13}	2.332043×10^{-13}	1.528101	1.665939×10^{-26}	1.393532×10^{-26}	6.473536
	GM_{B_1}	3.135621×10^{-15}	1.75465×10^{-13}	1.60281	1.785477×10^{-28}	1.485997×10^{-26}	6.800356
	GM_{B_2}	2.363672×10^{-14}	1.821614×10^{-13}	1.87499	1.252938×10^{-31}	2.946273×10^{-28}	7.49621
	GM_{B_3}	2.889434×10^{-14}	2.05878×10^{-13}	1.907362	1.616665×10^{-27}	1.965042×10^{-26}	7.697237
	GM_{B_4}	3.260876×10^{-13}	2.839510×10^{-13}	1.736964	1.791507×10^{-26}	1.020859×10^{-26}	6.22647

Table 9. In Proposed Method: Discrete WRM-SCPs, comparison of N-limited terms and various inputs of approximation integral parts R and discrete inner product terms M based on residual error functions, least square error, and running time.

The Methods		$(N,M,R)=(3,4,25)$			$(N,M,R)=(5,8,100)$		
		$LSE.$	$LSE_u.$	$R.Time$	$LSE.$	$LSE_u.$	$R.Time$
SC Ps	TM_{B1}	4.909790×10^{-15}	2.101762×10^{-13}	1.399527	8.029057×10^{-28}	1.209287×10^{-26}	6.11561
	TM_{B2}	1.17735×10^{-14}	2.08587×10^{-13}	1.40194	5.742859×10^{-28}	1.203215×10^{-26}	6.81247
	TM_{B3}	1.648105×10^{-14}	2.00788×10^{-13}	1.41070	1.128958×10^{-27}	1.190043×10^{-26}	6.17207
	TM_{B4}	4.040922×10^{-14}	1.871097×10^{-13}	1.452077	4.461245×10^{-27}	1.154531×10^{-26}	5.97789
	GM_{B1}	3.244297×10^{-15}	1.815158×10^{-13}	1.80709	1.796169×10^{-28}	1.49379×10^{-26}	9.489056
	GM_{B2}	2.090377×10^{-14}	2.06455×10^{-13}	2.216313	3.138877×10^{-31}	4.50428×10^{-28}	10.44275
	GM_{B3}	2.936548×10^{-14}	2.076915×10^{-13}	2.27205	1.623575×10^{-27}	1.97228×10^{-26}	10.23357
	GM_{B4}	1.978866×10^{-13}	2.503529×10^{-13}	1.851598	1.559088×10^{-26}	1.043131×10^{-26}	7.930474

Test problem 3. Consider a multi-fractional order linear IFDE with variable coefficients on the closed bounded interval $[a,b]; a,b \in \mathbb{R}$:

$$\begin{aligned}{}_{a}^{C}D_{t}^{\alpha_1}u(t)+t^2u(t)&=t^2-t^2e^{t-a}\\ &-\lim_{L\to\infty}\sum_{k=0}^{L}\left[\frac{(t-a)^{k-\alpha_1+1}}{\Gamma(k-\alpha_1+2)}-\frac{\lambda\sin(t)\,(b-a)^{k-\beta_2+2}}{\Gamma(k-\beta_2+3)}-\frac{\lambda t(b-a)^{k-\beta_1+2}}{\Gamma(k-\beta_1+3)}\right.\\ &\left.+\frac{\lambda(b-a)^{k-\beta_1+4}}{(k-\beta_1+4)\Gamma(k-\beta_1+2)}\right]\\ &+\lambda\int_a^b\left[\sin(t)\,{}_{a}^{C}D_{s}^{\beta_2}u(s)+(t-(s-a)^2)\,{}_{a}^{C}D_{s}^{\beta_1}u(s)\right]ds\end{aligned}$$

For all α_1,β_2,β_1 are real fractional order lies in (0,1] and $\lambda \in \mathbb{R}$ with boundary conditions: $B_1: u(a)-u(b)=e^{b-a}-1$, $B_2: u(a)-2u(b)=2\left(e^{b-a}-1\right)$. While the exact solution is $u(t)=1-e^{t-a}$.

Here, its numerical computation on the bounded interval $[a,b]=[1,2]$ is obtained using the proposed WRM. $N, M,$ and R, as well as the number of Mittag-Leffler [2], terms L, are used to determine the approximate solution values by the Tau and Galerkin methods for the given equation involving the fractional orders α_1, β_2 and β_1.

- Table 10 compares the time required to execute the algorithms for different input values using $(N,M,R)=(3,7,100)$ or $(5,7,100)$ and the approximate and exact solutions for fractional orders $(\alpha_1,\beta_2,\beta_1)=(0.6,0.4,0.3)$. Figures 4 and 5 display the error profile by comparing the approximate solutions and absolute errors for the LSE technique in SLPs and SCPs. All findings are displayed in Figures 4 and 5 and data in Table 10, while conforming to the boundary condition B_1. Since $\frac{g_{11}}{h_{11}}=-1$, the WR-Galerkin method's approximate solution depends on the specified boundary condition B_1. Therefore, we use equations (40), (47,50, and 52), and the approximate formula is given.
- Also, we create Table 11 using the approximate formula from equations (40) using equations (47,48, and 51) if the boundary condition B_2 is met because $\frac{g_{11}}{h_{11}} \neq -1$.

Table 10. $LSE.$, $LSE_u.$ and $R.Time/sec.$ of approximate solution for various values of $(\alpha_1, \beta_2, \beta_1) = (0.6, 0.4, 0.3)$ and eigenvalues λ and Mittag-Leffler terms L when $(M, R) = (7, 100)$ in test problem 3 on interval $[a, b] = [1,2]$ with B_1.

Input	Comparison	Proposed Method: Discrete WRM -SLPs TM	-SLPs GM	-SCPs TM	-SCPs GM	Quadrature Method Midpoint Type ([21])
$N = 3$ $\lambda = 1/2$ $L = 4$	$LSE.$	3.548948×10^{-06}	5.51067×10^{-06}	4.312582×10^{-06}	4.58469×10^{-06}	1.61675×10^{-04}
	$LSE_u.$	1.17968×10^{-04}	9.16654×10^{-05}	8.11279×10^{-05}	1.01599×10^{-04}	
	$R.Time/sec$	5.014019	4.036709	4.425734	3.98753	0.863048
$N = 5$ $\lambda = 1/2$ $L = 10$	$LSE.$	1.276209×10^{-11}	4.996115×10^{-11}	2.009769×10^{-11}	9.663105×10^{-12}	1.70487×10^{-04}
	$LSE_u.$	4.26875×10^{-10}	3.852717×10^{-09}	4.495085×10^{-10}	4.15475×10^{-10}	
	$R.Time/sec$	7.96736	46.69978	7.39776	7.89053	0.941413
$N = 5$ $\lambda = 6/5$ $L = 10$	$LSE.$	1.86674×10^{-11}	3.186251×10^{-11}	6.90652×10^{-11}	1.222962×10^{-11}	9.21954×10^{-05}
	$LSE_u.$	9.755279×10^{-10}	8.136239×10^{-10}	4.673885×10^{-10}	5.32667×10^{-10}	
	$R.Time/sec$	7.82818	8.02479	7.29219	7.755371	0.935585

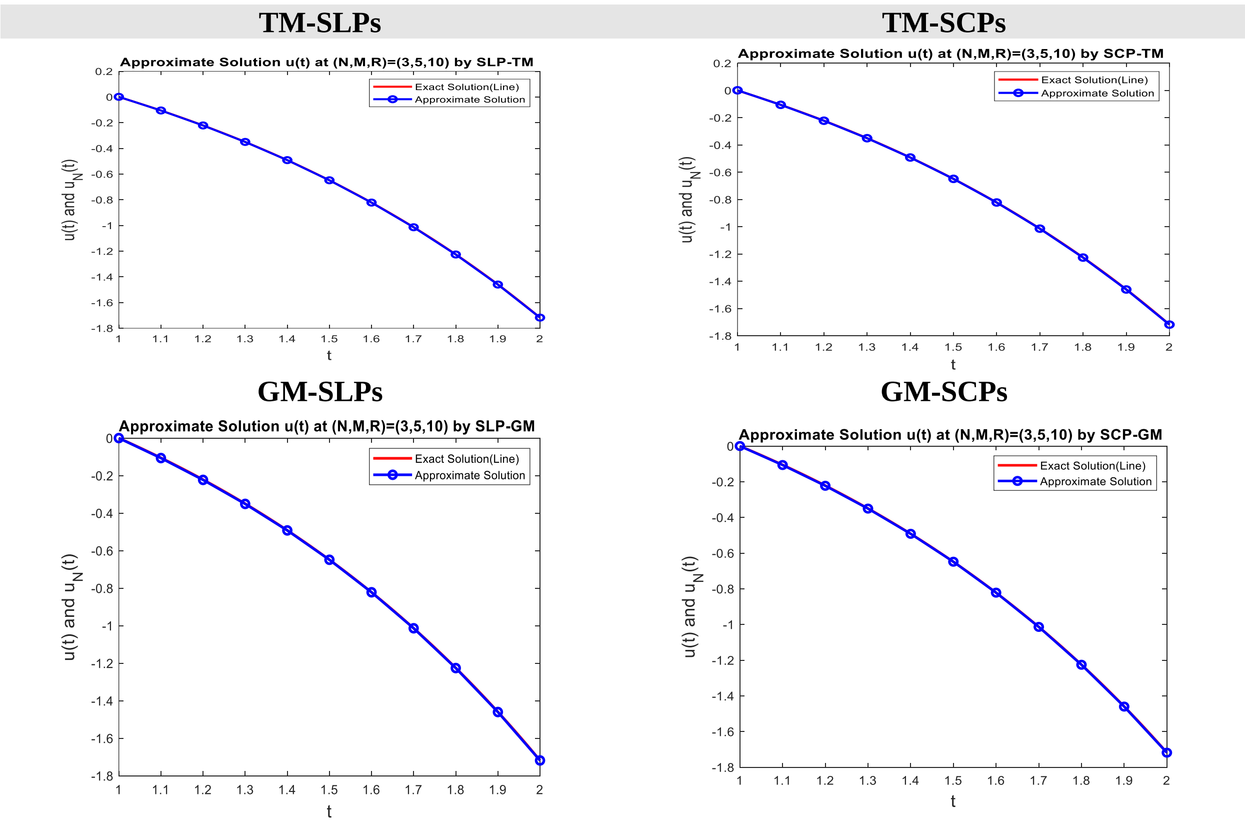

Fig.4. For test problem 3, by LSM with a step size of $h = 0.1$, the approximate solutions are shown by bullets, while the precise solution is shown by a solid line, $(N, M, R) = (3,5,10)$, $\lambda = 1/2$, $L = 4$.

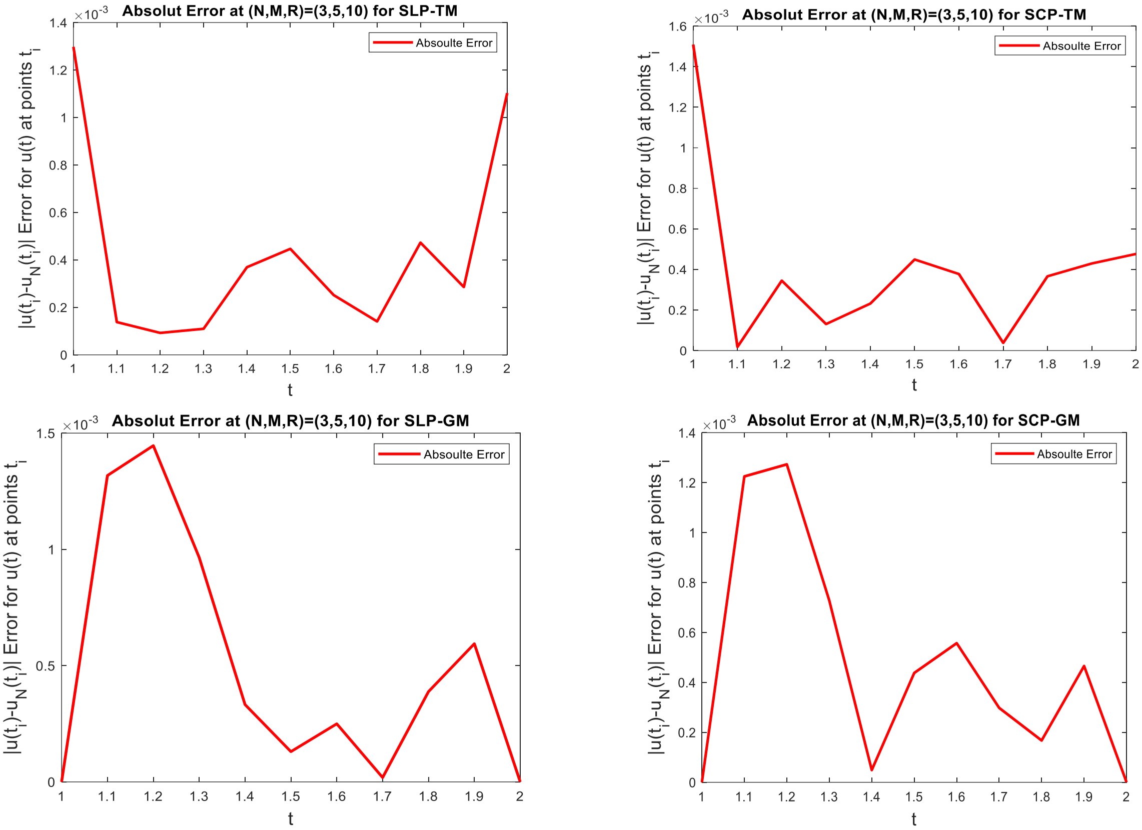

Fig. 5. Absolute error plot function $|u(t) - \hat{u}_N(t)|$ for $(N, M, R) = (3,5,10), h = 0.1$ for test problem 4, $\lambda = 1/2$, $L = 4$.

Table 11. $LSE.$, $LSE_u.$ and $R.Time/sec.$ of approximate solution for various values of $(\alpha_1, \beta_2, \beta_1) = (0.6,0.4,0.3)$ and eigenvalues λ and Mittag-Leffler terms L when $(M, R) = (7,100)$ in test problem 3 on the interval $[a, b] = [1,2]$ with B_2.

Input	Comparison	Proposed Method: Discrete WRM			
		-SLPs		-SCPs	
		TM	GM	TM	GM
$N = 3$ $\lambda = 1/2$ $L = 4$	$LSE.$	6.533163×10^{-06}	1.183218×10^{-05}	4.79890×10^{-06}	3.64799×10^{-06}
	$LSE_u.$	1.22853×10^{-04}	2.08558×10^{-04}	8.23389×10^{-05}	9.28291×10^{-05}
	$R.Time/sec$	4.14451	5.164886	3.96474	5.024212
$N = 5$ $\lambda = 1/2$ $L = 10$	$LSE.$	1.542612×10^{-11}	6.290095×10^{-11}	1.774429×10^{-11}	1.02960×10^{-11}
	$LSE_u.$	9.89554×10^{-10}	3.050017×10^{-09}	4.53821×10^{-10}	7.20839×10^{-10}
	$R.Time/sec$	6.83579	9.10266	7.19849	9.16469
$N = 5$ $\lambda = 6/5$ $L = 10$	$LSE.$	1.927177×10^{-11}	1.10162×10^{-10}	2.799847×10^{-11}	1.66499×10^{-11}
	$LSE_u.$	90874248×10^{-10}	3.068913×10^{-09}	4.534052×10^{-10}	7.28228×10^{-10}
	$R.Time/sec$	7.10366	9.12492	7.10638	8.98511

7. Conclusion

In general, multi-higher-order integro-fractional differential equations are difficult to solve exactly, require more mathematical work, or are nearly impossible to solve. A useful model for IFDEs (1)– (2) with non-constant coefficients is the Fredholm type, to which the discrete weighted residual approach (Tau and Galerkin) is generalized and implemented in this study. This process transforms the problem into a matrix-based linear algebraic system, which depends on the estimated coefficients. The method's ability to represent the result as a truncated orthogonal polynomial makes it simple to evaluate $u(t)$ for any value of t. This is a significant advantage. We need more terms from the approximate polynomial expression of functions in order to have the best approximation; in a sense, the truncation limit needs to be large enough. In discrete WRM, the number of terms must be sufficiently large. Special algorithms and MATLAB software were developed for each type of discrete WRM, and multiple examples are provided for illustration. The numerical results produced a good absolute error among the numerical approaches utilized in the literature.

The table with figures representing the running time and least square error for the accuracy and speed comparison leads to the following conclusions:

- In running test problems, the TM and GM algorithms have higher accuracy with equal step sizes than the trapezium, Simpson, and Midpoint methods ([21]).
- The accuracy of the results depends on both the process and the step length h; that is, when h is reduced, the accuracy increases, which raises the number of approximation terms N, dacite inner points M, and take partitions of numerical integrations R.
- In all of these approaches, the shifted-Legendre polynomial has better errors than the shifted-Chebyshev polynomial in the same N, and the solution converges to the precise solution with a very tiny least square error.

References

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Amsterdam, The Netherlands: Elsevier, 2006.

[2] I. Podlubny, *Fractional Differential Equations*. San Diego, CA, USA: Academic Press, 1999.

[3] R. M. Ganji, H. Jafari, and S. Nemati, “A new approach for solving integro-differential equations of variable order,” *J. Comput. Appl. Math.*, vol. 379, Art. no. 112946, Dec. 2020, doi: 10.1016/j.cam.2020.112946.

[4] K. O. Kareem, M. O. Olayiwola, M. O. Ogunniran, A. O. Asimiyu, A. O. Yunus, K. A. Adedokun, J. A. Adedeji, and A. I. Alaje, “Approximating higher order linear Fredholm integro-differential equations by an efficient Adomian decomposition method,” *Istanbul J. Math.*, vol. 2, no. 1, pp. 44–54, 2024, doi: 10.26650/ijmath.2024.00015.

[5] A. Abd-Elall, A. A. S. Zaghrout, K. R. Raslan, and K. K. Ali, “On the analytical and numerical study for nonlinear Fredholm integro-differential equations,” *Appl. Math. Inf. Sci.*, vol. 14, no. 5, pp. 921–929, 2020, doi: 10.18576/amis/140520.

[6] H. Hilmi, R. F. Mahmood, and S. S. Hama, “Existence and uniqueness of solution for boundary value problem of fractional order,” *Tikrit J. Pure Sci.*, vol. 29, no. 2, 2024, doi: 10.25130/tjps.v29i2.1562.

[7] A. A. El-Sayed, S. Boulaaras, and N. H. Sweilam, “Numerical solution of the fractional-order logistic equation via the first-kind Dickson polynomials and spectral tau method,” *Math. Methods Appl. Sci.*, vol. 46, no. 7, pp. 8004–8017, May 2023, doi: 10.1002/mma.7345.

[8] S. S. Ahmed and S. J. Mohammedfaeq, “Bessel collocation method for solving Fredholm–Volterra integro-fractional differential equations of multi-high order in the Caputo sense,” *Symmetry*, vol. 13, no. 12, Art. no. 2354, 2021, doi: 10.3390/sym13122354.

[9] E. Imamoglu, “Hermite interpolation with Dickson polynomials and Bernstein basis polynomials,” *Eskisehir Tech. Univ. J. Sci. Technol. B*, vol. 11, no. 2, pp. 158–166, 2023, doi: 10.20290/estubtdb.1279073.

[10] M. I. Syam and B. S. Attili, "Weighted residual method for obtaining positive solutions of two-point nonlinear boundary value problems," *Appl. Math. Comput.*, vol. 176, pp. 775–784, 2006, doi: 10.1016/j.amc.2005.10.026.
[11] S. S. Ahmed and M. R. Ahmed, "The weighted residual method in solution of more general nonlinear integro-fractional differential equation of Volterra–Hammerstein type with variable coefficients," *J. Zankoy Sulaimani A*, vol. 21, no. 1, pp. 127–144, Jun. 2019, doi: 10.17656/jzs.10751.
[12] I. A. Dhari and M. M. Mustafa, "Numerical solution of linear Volterra integral equation of the second kind with delay using Lagrange polynomials," *Iraqi J. Sci.*, vol. 65, no. 3, pp. 1541–1547, 2024, doi: 10.24996/ijs.2024.65.3.30.
[13] J. Mi and J. Huang, "Collocation method for solving two-dimensional nonlinear Volterra–Fredholm integral equations with convergence analysis," *J. Comput. Appl. Math.*, vol. 428, Art. no. 115188, Aug. 2023, doi: 10.1016/j.cam.2023.115188.
[14] Y. Liu, Z. Zhao, and Y. Zhang, "Approximate solutions to fractional differential equations," *Appl. Math. Mech. (Engl. Ed.)*, vol. 44, pp. 1791–1802, 2023, doi: 10.1007/s10483-023-3041-9.
[15] S. S. Ahmed and S. J. Mohammedfaeq, "Operational matrix of generalized block pulse function for solving fractional Volterra–Fredholm integro-differential equations," *J. Southwest Jiaotong Univ.*, vol. 57, pp. 39–59, 2022, doi: 10.35741/issn.0258-2724.57.3.4.
[16] M. R. Ahmed, S. S. Ahmed, and P. O. Sabir, "Approximate solution to the system of nonlinear Volterra integro-fractional differential equations with variable coefficients using linear B-spline functions," *J. Southwest Jiaotong Univ.*, vol. 59, no. 4, 2024, doi: 10.35741/issn.0258-2724.59.4.26.
[17] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*. Singapore: World Scientific, 2022.
[18] S. A. HamaSalih, *Some Computational Methods for Solving Linear Volterra Integro-Fractional Differential Equations*, M.Sc. thesis, Univ. of Sulaimani, Sulaymaniyah, Iraq, 2011.
[19] D. C. Zahir, *Numerical Solutions for the Most General Multi-Higher Fractional Order Linear Integro-Differential Equations of Fredholm Type in Caputo Sense*, M.Sc. thesis, Univ. of Sulaimani, Sulaymaniyah, Iraq, 2017.
[20] P. Grandclément and J. Novak, "Spectral methods for numerical relativity," *Living Rev. Relativ.*, vol. 12, no. 1, 2009, doi: 10.12942/lrr-2009-1.
[21] D. C. Zahir, S. S. Ahmed, and S. J. Mohammedfaeq, "A midpoint quadratic approach for solving numerically multi-order fractional integro-differential equation," *J. Univ. Babylon Pure Appl. Sci.*, vol. 33, no. 3, 2025, doi: 10.29196/jubpas.v33i3.5974.
[22] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*. New York, NY, USA: Dover Publications, 2000.
[23] J. Shen, "Efficient spectral-Galerkin methods III: Polar and cylindrical geometries," *SIAM J. Sci. Comput.*, vol. 18, no. 6, pp. 1583–1604, 1997, doi: 10.1137/S1064827595295301.

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الطريقة المتقطعة للبواقي الموزونة باستخدام تقنيتي تاو وجالركن لحل المعادلات التكاملية-التفاضلية ذات الرتب الكسرية العليا المتعددة من نوع فريدهولم

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الكلمات المفتاحية

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الملخص

هدف هذه المقالة هو ابتكار وتنفيذ طرق جديدة لاستخدام كثيرات الحدود المحوّلة لليجيندر وتشبيشيف لحل المعادلات التكاملية-التفاضلية من نوع فريدهولم ذات الرتب الكسرية المتعددة .سيتم استخدام طريقتي تاو وجالركن بالاقتران مع الطريقة المتقطعة للبواقي الموزونة .كما ستُستخدم صيغة كلينشو-كورتيس التربيعية حسابيًا لتقييم الحدود التكاملية.

من أجل تحديد معاملات المتعامدات للحلول التقريبية، تعتمد هذه الدراسة على مصفوفة تشغيل لتحويل معادلات فريدهولم ذات الرتب الكسرية إلى نظام من المعادلات الجبرية الخطية .تمكّن هذه الطريقة من تطوير خوارزميات لكل من الطريقتين للحصول على حلول تقريبية للمعادلة.

بالإضافة إلى ذلك، يتم توضيح والتحقق من صحة وكفاءة هذه التقنية، ومقارنتها مع نتائج سابقة باستخدام أمثلة عددية .يتم تنفيذ معظم التطبيقات على جهاز حاسوب يحتوي على برنامج ماتلاب الإصدار 9.7.

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