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A Generalization of The Baskakov-Kantorovich Sequence Based on Two Nonnegative Parameters

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A B S T R A C T

This paper introduces a generalized sequence of linear positive operators, denoted the (r, α) -Baskakov-Kantorovich sequence, which extends the classical Baskakov-Kantorovich sequence by introducing two nonnegative parameters, r and α. These parameters enhance the flexibility of this sequence, enabling finer control over the approximation process. Convergence of the sequence to any continuous function on nonnegative real numbers is established using Korovkin's theorem. Furthermore, a Voronovsky-type asymptotic formula is derived, providing insight into the order of convergence as η tends to infinity. To support the theoretical results, a numerical example is presented in which the sequence is applied to approximate a test function. The numerical findings indicate that the proposed operators not only improve approximation accuracy but also reveal flexibility in handling different types of parameterized test functions, underscoring their potential as a more versatile and effective tool in approximation theory. This study contributes to the development of parametric generalizations of classical operators with enhanced approximation properties.

1. Introduction

Approximating continuous and discontinuous functions has been a central problem in mathematical analysis since ancient times, as mathematicians work to create operators that can more accurately simulate the complicated behavior of functions. In this context, several mathematicians made foundations contributions that established the theory of linear positive operators. Among them was Baskakov [1], who introduced a new formulation of approximation operators that later came to be known as the classical Baskakov sequence used to approximate continuous real-valued functions on the interval $[0, \infty)$. This sequence of operators defined as

$$\mathcal{B}_\eta(f; z) = \sum_{\kappa=0}^{\infty} \wp_{\eta,\kappa}(z)\, f\left(\frac{\kappa}{\eta}\right), z \in [0, \infty), \eta \in \mathbb{N}, \tag{1}$$

where

$$\wp_{\eta,\kappa}(z) = \frac{z^\kappa}{(1+z)^{\eta+\kappa}} \binom{\eta + \kappa - 1}{\kappa}, \tag{2}$$

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for a function $f \in C_\beta[0,\infty) := \{f \in C[0,\infty): |f(t)| \le M(1+t)^\beta, \text{for some } M > 0, \beta > 0\}$ and any $z \in [0,\infty)$. However, this kind of sequence is not suitable for discontinuous functions. Ditzian and Totik [2] introduced the Baskakov-Kantorovich sequence for the integrable function space as

$$\mathrm{L}_\eta(f;z) = \eta \sum_{\kappa=0}^{\infty} \mathscr{p}_{\eta,\kappa}(z) \int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}} f(t)\, dt,$$

where $\mathscr{p}_{\eta,\kappa}(z)$ is defined in (2). The classical Baskakov sequence, Baskakov-Kantorovich sequence, and some of their generalizations have been the subject of extensive research. Gupta [3] introduced a modification of the Baskakov sequence by using the Beta function. Abel and Gupta [4] proposed a Bézier version of the Baskakov-Kantorovich sequence. Mohammad and Abdul-Rezaq [5] generalized a sequence of Beta operators depending on a nonnegative integer. Chen et al. [6] constructed a generalization of the sequence of Bernstein operators based on a nonnegative real parameter. İnce İlarslan et al. [7] defined a generalization sequence of the Baskakov-Kantorovich operators based on a nonnegative parameter. Yılmaz et al. [8] introduced a sequence of Baskakov-Kantorovich type operators that preserve the exponential function. Mohammad et al. [9] developed a family of sequences of Baskakov-type operators parameterized $s > -1/2$. Mohiuddine et al. [10] defined a sequence of Baskakov-Kantorovich Stanc-type operators involving a real parameter. Rao and Rani [11] presented a sequence of Baskakov-Kantorovich operators depending on two parameters to approximate a class of Lebesgue measurable functions on [0, ∞). Kajla et al. [12] constructed a sequence of Baskakov-Jain type operators involving two real parameters. Cheng and Mohiuddine [13] modified a sequence of the Baskakov operators using the second central moment of the classical Baskakov operators. Goyal and Agrawal [14] extended the Baskakov-Kantorovich sequence to two dimensions and assessed its degree of approximation. Rao and Malik [15] introduced a family of hybrid operators depending on a real parameter. Jasim and Mohammad [16] proposed a generalization of the multiple sums of sequences based on parameter $s > -1/2$. Aktuglu et al. [17] modified a sequence of Baskakov–Kantorovich operators by incorporating a function ψ, which led to a significant improvement.

2. $(\mathscr{r},\alpha)$-Baskakov-Kantorovich sequence

İnce İlarslan et al. [7] introduced the α-Baskakov-Kantorovich sequence of the operators defined in (1), as:

$$L_\eta^\alpha(f;z) = \eta \sum_{\kappa=0}^{\infty} \mathscr{p}_{\eta,\kappa}^\alpha(z) \int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}} f(t)\, dt,$$

where $\alpha \in [0,1], f \in C_\beta[0,\infty)$ and

$$\mathscr{p}_{\eta,\kappa}^\alpha(z) = \frac{z^{\kappa-1}}{(z+1)^{\eta+\kappa}}\left(\frac{\alpha z}{(1+z)}\binom{\eta+\kappa-1}{\kappa} - (1-\alpha)(1+z)\binom{\eta+\kappa-3}{\kappa-2} + (1-\alpha)z\binom{\eta+\kappa-1}{\kappa}\right).$$

In this article, the generalization of the Baskakov-Kantorovich sequence is introduced as:

$$K_\eta^{\mathscr{r},\alpha}(f;z) = \frac{\eta}{(\eta)_{\mathscr{r}}} \sum_{\kappa=0}^{\infty} b_{\eta,\kappa}^{\mathscr{r},\alpha}(z) \int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}} f(t)\, dt, \tag{3}$$

where $\mathscr{r} \in N^0 := \{0,1,2,\dots\}$, $(\eta)_{\mathscr{r}} = \prod_{i=0}^{\mathscr{r}-1}(\eta+i) = \frac{(\eta+\mathscr{r}-1)!}{(\eta-1)!}$ and

$$b_{\eta,\kappa}^{\mathscr{r},\alpha}(z) = \frac{z^{\kappa-1}}{(1+z)^{\eta+\mathscr{r}+\kappa-1}}\Bigg(\alpha(\eta+\kappa)_{\mathscr{r}}\frac{z}{(1+z)}\binom{\eta+\kappa-1}{\kappa} - (1-\alpha)(\eta+\kappa-2)_{\mathscr{r}}(1+z)\binom{\eta+\kappa-3}{\kappa-2} + (1-\alpha)(\eta+\kappa)_{\mathscr{r}}\, z\binom{\eta+\kappa-1}{\kappa}\Bigg).$$

This generalization is called $(\mathscr{r},\alpha)$-Baskakov-Kantorovich sequence.

Note that in (3) the classical Baskakov-Kantorovich sequence can be obtained by choosing $\mathscr{r} = 0$ and $\alpha = 1$.

3. Preliminary Results

Here, some preliminary results for the sequence $K_\eta^{\mathscr{r},\alpha}(f;z)$ are established in this study.

Lemma 3.1. For $c_1 \in \{0,1\}$ the functions
$u_{\eta+c_1,\kappa}^{\mathscr{r}}(z) = \binom{\eta + c_1 + \mathscr{r} + \kappa - 2}{\kappa} \frac{z^\kappa}{(1+z)^{\eta+c_1+\mathscr{r}+\kappa-1}}$, the following facts hold

$$z(1+z)\frac{d}{dz}u_{\eta+c_1,\kappa}^{\mathscr{r}}(z) = (\kappa + (-\eta - c_1 - \mathscr{r} + 1)z)u_{\eta+c_1,\kappa}^{\mathscr{r}}(z).$$

Proof. By derivativeing of the function $u_{\eta+c_1,\kappa}^{\mathscr{r}}(z)$ and multiplication of both sides by $z(1+z)$, one gets

$$\begin{aligned} z(1+z)\frac{d}{dz}u_{\eta+c_1,\kappa}^{\mathscr{r}}(z) &= \binom{\eta + c_1 + \mathscr{r} + \kappa - 2}{\kappa} \frac{z^\kappa}{(1+z)^{\eta+c_1+\mathscr{r}+\kappa-1}} \\ &\quad \times (\kappa(1+z) + (-\eta - c_1 - \mathscr{r} - \kappa + 1)z) \\ &= (\kappa + (-\eta - c_1 - \mathscr{r} + 1)z)u_{\eta+c_1,\kappa}^{\mathscr{r}}(z)\,. \blacksquare \end{aligned}$$

Lemma 3.2. For $m \in N^0 := \{0,1,2,\ldots\}$ and $c_1, c_2, j \in \{0,1\}$ the function

$$\psi_{m,\eta+c_1,c_{2,j}}(z) = \sum_{\kappa=0}^{\infty} \left(\frac{\kappa + c_2 + j}{\eta} - z\right)^m \left(1 - c_2 + \frac{\kappa}{\eta + \mathscr{r} - 1}\right)^{1-c_1} u_{\eta+c_1,\kappa}^{\mathscr{r}}(z) \tag{4}$$

has the following properties

(i) $\psi_{0,\eta+c_1,c_{2,j}}(z) = \sum_{\kappa=0}^{\infty} \left(1 - c_2 + \frac{\kappa}{\eta + \mathscr{r} - 1}\right)^{1-c_1} u_{\eta+c_1,\kappa}^{\mathscr{r}}(z),$

(ii) $\psi_{1,\eta+c_1,c_2,j}(z) = \sum_{\kappa=0}^{\infty} \left(\frac{\kappa + c_2 + j}{\eta}\right) \left(1 - c_2 + \frac{\kappa}{\eta + \mathscr{r} - 1}\right)^{1-c_1} u_{\eta+c_1,\kappa}^{\mathscr{r}}(z) - z\psi_{0,\eta+c_1,c_2,j}(z),$

(iii)
$$\begin{aligned} \psi_{m+1,\eta+c_1,c_2,j}(z) &= \frac{1}{\eta}\Bigg((z^2+z)\left(\frac{d}{dz}\psi_{m,\eta+c_1,c_2,j}(z) + m\psi_{m-1,\eta+c_1,c_2,j}(z)\right) \\ &\quad + \big((-\mathscr{r} - c_1 + 1)z - c_2 - j\big)\psi_{m,\eta+c_1,c_2,j}(z)\Bigg). \end{aligned}$$

Proof. The proofs of properties (i)-(ii) are straightforward.
(iii) By the derivative of both sides of equation (4), the result is

$$\begin{aligned} \frac{d}{dz}\psi_{m,\eta+c_1,c_2,j}(z) &= \sum_{\kappa=0}^{\infty} \left(\frac{\kappa + c_2 + j}{\eta} - z\right)^m \left(1 - c_2 + \frac{\kappa}{\eta + \mathscr{r} - 1}\right)^{1-c_1} \frac{d}{dz}u_{\eta+c_1,\kappa}^{\mathscr{r}}(z) \\ &\quad - m\sum_{\kappa=0}^{\infty} \left(\frac{\kappa + c_2 + j}{\eta} - z\right)^{m-1} \left(1 - c_2 + \frac{\kappa}{\eta + \mathscr{r} - 1}\right)^{1-c_1} u_{\eta+c_1,\kappa}^{\mathscr{r}}(z) \\ &= \sum_{\kappa=0}^{\infty} \left(\frac{\kappa + c_2 + j}{\eta} - z\right)^m \left(1 - c_2 + \frac{\kappa}{\eta + \mathscr{r} - 1}\right)^{1-c_1} \frac{d}{dz}u_{\eta+c_1,\kappa}^{\mathscr{r}}(z) - m\psi_{m-1,\eta+c_1,c_2,j}(z). \end{aligned}$$

Now, multiplying both sides by (z^2+z) and applying Lemma 3.1, the outcome is

$$(z^2+z)\left(\frac{d}{dz}\psi_{m,\eta+c_1,c_2,j}(z) + m\psi_{m-1,\eta+c_1,c_2,j}(z)\right)$$

$$= \sum_{\kappa=0}^{\infty} \left(\frac{\kappa + c_2 + j}{\eta} - z\right)^m \left(1 - c_2 + \frac{\kappa}{\eta + r - 1}\right)^{1-c_1} (\kappa + (-\eta - c_1 - r + 1)z) u_{\eta+c_1,\kappa}^{r}(z)$$
$$= \sum_{\kappa=0}^{\infty} \left(\frac{\kappa + c_2 + j}{\eta} - z\right)^m \left(1 - c_2 + \frac{\kappa}{\eta + r - 1}\right)^{1-c_1}$$
$$\times \left(\eta\left(\frac{\kappa + c_2 + j}{\eta} - z\right) + (-r - c_1 + 1)z - c_2 - j\right) u_{\eta+c_1,\kappa}^{r}(z)$$
$$= \eta\psi_{m+1,\eta+c_1,c_2,j}(z) + \left((-r - c_1 + 1)z - c_2 - j\right)\psi_{m,\eta+c_1,c_2,j}(z).$$

Hence,

$$\psi_{m+1,\eta+c_1,c_2,j}(z) = \frac{1}{\eta}\left((z^2 + z)\left(\frac{d}{dz}\psi_{m,\eta+c_1,c_2,j}(z) + m\psi_{m-1,\eta+c_1,c_2,j}(z)\right) + \left((-r - c_1 + 1)z - c_2 - j\right)\psi_{m,\eta+c_1,c_2,j}(z)\right). \blacksquare$$

Lemma 3.3. For $m \in \mathbb{N}^0$the function

$$\emptyset_{m,\eta,j}(z) = \frac{1}{(\eta)_r} \sum_{\kappa=0}^{\infty} b_{\eta,\kappa}^{r,\alpha}(z) \left(\frac{\kappa + j}{\eta} - z\right)^m$$

has the following properties

(i) $\emptyset_{0,\eta,j}(z) = 1,$

(ii) $\emptyset_{1,\eta,j}(z) = \frac{1}{\eta}(\eta + 2\alpha + r - 2)z + \left(\frac{j}{\eta} - z\right),$

(iii) $\emptyset_{m+1,\eta,j}(z) = (1 - \alpha)\left(\psi_{m,\eta,0,j}(z) - \psi_{m,\eta,1,j}(z)\right) + \psi_{m,\eta+1,0,j}(z).$

Proof. Proofs for properties (i) and (ii) are acquired directly, and the proof of (iii) can be acquired as follows

$$\emptyset_{m,\eta,j}(z) = \frac{1}{(\eta)_r} \sum_{\kappa=0}^{\infty} b_{\eta,\kappa}^{r,\alpha}(z) \left(\frac{\kappa + j}{\eta} - z\right)^m$$
$$= (1 - \alpha) \sum_{\kappa=0}^{\infty} \left(\left(\frac{\kappa + j}{\eta} - z\right)^m \left(1 + \frac{\kappa}{\eta + r - 1}\right) - \left(\frac{\kappa + 1 + j}{\eta} - z\right)^m \left(\frac{\kappa}{\eta + r - 1}\right)\right) u_{\eta,\kappa}^{r}(z)$$
$$+ \alpha \sum_{\kappa=0}^{\infty} \left(\frac{\kappa + j}{\eta} - z\right)^m u_{\eta+1,\kappa}^{r}(z)$$
$$= (1 - \alpha)\left(\psi_{m,\eta,0,j}(z) - \psi_{m,\eta,1,j}(z)\right) + \alpha\psi_{m,\eta+1,0,j}(z). \blacksquare$$

Lemma 3.4. The m-th order moments for the sequence $K_\eta^{r,\alpha}(f; z)$ is defined by

$$M_{\eta,m}^{r,\alpha}(z) = K_\eta^{r,\alpha}((t - z)^m; z) = \frac{\eta}{(\eta)_r} \sum_{\kappa=0}^{\infty} b_{\eta,\kappa}^{r,\alpha}(z) \int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}} (t - z)^m \, dt,$$

and has the properties listed below:

(i) $M_{\eta,0}^{r,\alpha}(z) = 1,$

(ii) $M_{\eta,1}^{r,\alpha}(z) = \frac{1}{\eta}(2\alpha + r - 2) + \frac{1}{2\eta},$

$$(iii)\ M_{\eta,2}^{r,\alpha}(z) = \frac{4\alpha z^2 + 2rz^2 - 3z^2 + 2z - 2z(2\alpha + r - 2)}{\eta} + \frac{4\alpha r z^2 + r^2 z^2 - 3rz^2 + 4\alpha z + rz + 2\alpha + r - 4z - \frac{5}{3}}{\eta^2},$$

$$(iv)\ M_{\eta,m}^{r,\alpha}(z) = \frac{\eta}{(m+1)}\left(\emptyset_{m+1,\eta,1}(z) - \emptyset_{m+1,\eta,0}(z)\right).$$

Proof. The proof of properties (i)-(iii) are acquired directly, and the proof of (iv) can be acquired below

$$M_{\eta,m}^{r,\alpha}(z) = \frac{\eta}{(\eta)_r}\sum_{\kappa=0}^{\infty} b_{\eta,\kappa}^{r,\alpha}(z)\int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}} (t-z)^m\, dt$$
$$= \frac{\eta}{(m+1)}\sum_{\kappa=0}^{\infty} b_{\eta,\kappa}^{r,\alpha}(z)\left(\left(\frac{\kappa+1}{\eta} - z\right)^{m+1} - \left(\frac{\kappa}{\eta} - z\right)^{m+1}\right).$$

By using Lemma 3.3, one gets

$$M_{\eta,m}^{r,\alpha}(z) = \frac{\eta}{(m+1)}\left(\emptyset_{m+1,\eta,1}(z) - \emptyset_{m+1,\eta,0}(z)\right). \blacksquare$$

Lemma 3.5. The function $M_{\eta,m}^{r,\alpha}(z)$, has the following facts:

(i) $\lim_{\eta\to\infty} \eta\, M_{\eta,0}^{r,\alpha}(z) = 1,$

(ii) $\lim_{\eta\to\infty} \eta\, M_{\eta,1}^{r,\alpha}(z) = (2\alpha + r - 2) + \frac{1}{2},$

(iii) $\lim_{\eta\to\infty} \eta\, M_{\eta,2}^{r,\alpha}(z) = 4\alpha z^2 + 2rz^2 - 3z^2 + 2z - 2z(2\alpha + r - 2),$

(iv) $\lim_{\eta\to\infty} \eta\, M_{\eta,m}^{r,\alpha}(z) = 0, where\ m \geq 3.$

Proof. By direct, the proof of this lemma holds. ■

Definition 3.1. For $\sigma > 0$, the modulus of continuity $\varphi_\sigma(f)$ is defined by

$$\varphi_\sigma(f) = \sup_{\substack{|t-z|\leq\sigma \\ t,z\in[0,\infty)}} |f(t) - f(z)|.$$

4. Main Results

This section presents the convergence theorem and Voronovskaja-type asymptotic theorem for the (r, α)-Baskakov-Kantorovich sequence, and the theorem for error estimation using the modulus of continuity.

Theorem 4.1. For $f \in C_\beta[0,\infty)$, the sequence $K_\eta^{r,\alpha}(f; z)$ is converged to the function f as $\eta \to \infty$.

Proof. By using Korovkin's theorem [18], the proof of the above theorem comes.

$$(i)\ K_\eta^{r,\alpha}(1; z) = \frac{\eta}{(\eta)_r}\sum_{\kappa=0}^{\infty} b_{\eta,\kappa}^{r,\alpha}(z)\int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}} dt = 1,$$

$$(ii)\ K_\eta^{r,\alpha}(t; z) = \frac{\eta}{(\eta)_r}\sum_{\kappa=0}^{\infty} b_{\eta,\kappa}^{r,\alpha}(z)\int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}} t\, dt = z + \frac{1}{\eta}(2\alpha + r - 2) + \frac{1}{2\eta},$$

$$(iii)\ K_\eta^{r,\alpha}(t^2; z) = \frac{\eta}{(\eta)_r}\sum_{\kappa=0}^{\infty} b_{\eta,\kappa}^{r,\alpha}(z)\int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}} t^2\, dt = z^2 + \frac{4\alpha z^2 + 2rz^2 - 3z^2 + 2z}{\eta}$$

$$+\frac{4\alpha r z^2 + r^2 z^2 - 3r z^2 + 4\alpha z + rz + 2\alpha + r - 4z - \frac{5}{3}}{\eta^2}.$$

Hence, $K_\eta^{r,\alpha}(f;z) \to f(z)$ as $\eta \to \infty$. ■

Theorem 4.2. For $f \in C_\beta[0,\infty)$ and f'' exists and is continuous then,

$$\lim_{\eta\to\infty} \eta(K_\eta^{r,\alpha}(f;z) - f(z))$$
$$= \left((2\alpha + r - 2) + \frac{1}{2}\right) f'(z) + \frac{1}{2}\left(4\alpha z^2 + 2r z^2 - 3z^2 + 2z - 2z(2\alpha + r - 2)\right) f''(z).$$

Proof. By using Tylor's expansion of $f(t)$, the result is

$$f(t) = f(z) + (t-z) f'(z) + \frac{(t-z)^2}{2!} f''(z) + \varepsilon(t,z)(t-z)^2,$$

where $\varepsilon(t,z) \to 0$ as $t \to z$.

Now, by taking the sequence $K_\eta^{r,\alpha}(.;z)$ for both sides and using Lemma 3.4, one gets

$$K_\eta^{r,\alpha}(f(t);z) = f(z) K_\eta^{r,\alpha}(1;z) + f'(z) M_{\eta,1}^{r,\alpha}(z) + \frac{1}{2!} f''(z) M_{\eta,2}^{r,\alpha}(z) + K_\eta^{r,\alpha}((t-z)^2 \varepsilon(t,z);z).$$

Then,

$$\lim_{\eta\to\infty} \eta\left(K_\eta^{r,\alpha}(f(t);z) - f(z)\right)$$
$$= f'(z) \lim_{\eta\to\infty}\left(\eta M_{\eta,1}^{r,\alpha}(z)\right) + \frac{1}{2!} f''(z) \lim_{\eta\to\infty}\left(\eta M_{\eta,2}^{r,\alpha}(z)\right) + \lim_{\eta\to\infty}\left(\eta K_\eta^{r,\alpha}((t-z)^2 \varepsilon(t,z);z)\right).$$

By using Lemma 3.5, the outcome is

$$\lim_{\eta\to\infty} \eta\left(K_\eta^{r,\alpha}(f(t);z) - f(z)\right)$$
$$= \left((2\alpha + r - 2) + \frac{1}{2}\right) f'(z) + \frac{1}{2}\left(4\alpha z^2 + 2r z^2 - 3z^2 + 2z - 2z(2\alpha + r - 2)\right) f''(z)$$
$$+ \lim_{\eta\to\infty}\left(\eta K_\eta^{r,\alpha}((t-z)^2 \varepsilon(t,z);z)\right).$$

Now, if it shows that the $\lim_{\eta\to\infty}\left(\eta K_\eta^{r,\alpha}((t-z)^2 \varepsilon(t,z);z)\right) = 0$, then the proof is done.

$$K_\eta^{r,\alpha}(\eta (t-z)^2 \varepsilon(t,z);z) = \frac{\eta^2}{(\eta)_r} \sum_{\kappa=0}^{\infty} \left(b_{\eta,\kappa}^{r,\alpha}(z)\right)^{\frac{1}{2}+\frac{1}{2}} \int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}} (t-z)^2 \varepsilon(t,z)\, dt.$$

By the Cauchy-Schwarz inequality, the result is

$$K_\eta^{r,\alpha}(\eta \varepsilon(t,z)(t-z)^2;z) \le \eta \left(\frac{\eta}{(\eta)_r} \sum_{\kappa=0}^{\infty} b_{\eta,\kappa}^{r,\alpha}(z) \int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}} (t-z)^4\, dt\right)^{\frac{1}{2}}$$
$$\times \left(\frac{\eta}{(\eta)_r} \sum_{\kappa=0}^{\infty} b_{\eta,\kappa}^{r,\alpha}(z) \int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}} \left(\varepsilon(t,z)\right)^2 dt\right)^{\frac{1}{2}}.$$

Then,

$$\lim_{\eta\to\infty}\left(\eta K_\eta^{r,\alpha}((t-z)^2 \varepsilon(t,z);z)\right) \le \lim_{\eta\to\infty} \sqrt{\eta}\left(\eta M_{\eta,4}^{r,\alpha}(z)\right)^{\frac{1}{2}} \times \left(K_\eta^{r,\alpha}\left(\left(\varepsilon(t,z)\right)^2;z\right)\right)^{\frac{1}{2}}.$$

By using Korovkin's theorem [18] and Lemma 3.5, the outcome is

$K_\eta^{r,\alpha}\left(\left(\varepsilon(t,z)\right)^2;z\right) = \left(\varepsilon(z,z)\right)^2 = 0$ as $\eta \to \infty$ and $\lim_{\eta\to\infty} \eta\, M_{\eta,4}^{r,\alpha}(z) = 0$.

Then,

$$\lim_{\eta\to\infty}\left(\eta K_\eta^{r,\alpha}((t-z)^2\varepsilon(t,z);z)\right)=0.$$

Hence,

$$\lim_{\eta\to\infty}\eta\left(K_\eta^{r,\alpha}(f;z)-f(z)\right)$$
$$=\left((2\alpha+r-2)+\frac{1}{2}\right)f'(z)$$
$$+\frac{1}{2}\left(4\alpha z^2+2rz^2-3z^2+2z-2z(2\alpha+r-2)\right)f''(z).\ \blacksquare$$

Theorem 4.3. For $f\in C_\beta[0,\infty)$, then

$$\left|K_\eta^{r,\alpha}(f;z)-f(z)\right|\le 2\varphi_\sigma(f),$$
$$where\ \sigma=\sqrt{M_{\eta,2}^{r,\alpha}(z)}.$$

Proof. From the property of the modulus of continuity, the result is:

$$|f(t)-f(z)|\le\varphi_\sigma(f)\left(\frac{|t-z|}{\sigma}+1\right).$$

Now, take the sequence $K_\eta^{r,\alpha}(f;z)$ for both sides, the outcome is

$$\left|K_\eta^{r,\alpha}(f(t);z)-f(z)\right|\le\varphi_\sigma(f)\left(\frac{K_\eta^{r,\alpha}(|t-z|;z)}{\sigma}+1\right)$$
$$=\varphi_\sigma(f)\left(\left(\frac{\eta}{\sigma(\eta)_r}\sum_{\kappa=0}^{\infty}\left(b_{\eta,\kappa}^{r,\alpha}(z)\right)^{\frac{1}{2}+\frac{1}{2}}\int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}}|t-z|\,dt\right)+1\right).$$

By using the Cauchy-Schwarz inequality and Lemma 3.4, one has

$$\left|K_\eta^{r,\alpha}(f(t);z)-f(z)\right|\le\frac{\varphi_\sigma(f)}{\sigma}\left(\frac{\eta}{(\eta)_r}\sum_{\kappa=0}^{\infty}b_{\eta,\kappa}^{r,\alpha}(z)\int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}}|t-z|^2\,dt\right)^{\frac{1}{2}}$$
$$\times\left(\frac{\eta}{(\eta)_r}\sum_{\kappa=0}^{\infty}b_{\eta,\kappa}^{r,\alpha}(z)\int_{\frac{\kappa}{\eta}}^{\frac{\kappa+1}{\eta}}dt\right)^{\frac{1}{2}}+\varphi_\sigma(f)$$
$$=\varphi_\sigma(f)\left(\frac{1}{\sigma}\left(M_{\eta,2}^{r,\alpha}(z)M_{\eta,0}^{r,\alpha}(z)\right)^{\frac{1}{2}}+1\right)$$

Since, $\sqrt{M_{\eta,0}^{r,\alpha}(z)}=1.$

Hence,

$$\left|K_\eta^{r,\alpha}(f;z)-f(z)\right|\le 2\varphi_\sigma(f).\ \blacksquare$$

5. Numerical Example

This section presents a numerical example of a test function, and it has been shown that the numerical results of the sequence change relatively with variations in the values of the two parameters r and α. This behavior is illustrated through graphical representations, in addition to calculating the average error rate for different values of η.

Example 5.1: $f(z)=\sin(5z)e^{-z}$, $z\in[0,4]$.

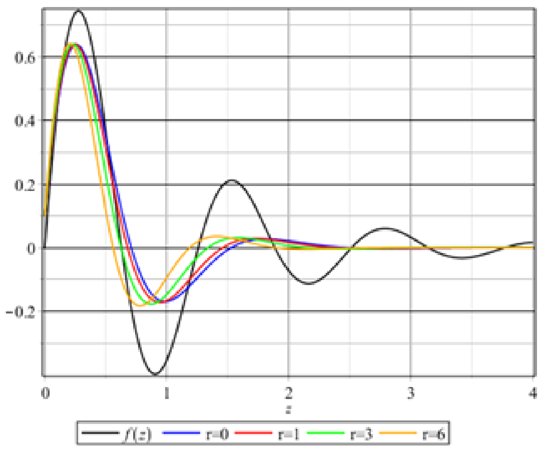

Fig. 1. The numerical convergence of the sequence $K_{\eta}^{r,\alpha}$ to the function $f(z)$, with $\eta = 25$ when $\alpha = 0.2$ and $r = 0, 1, 3, 6$.

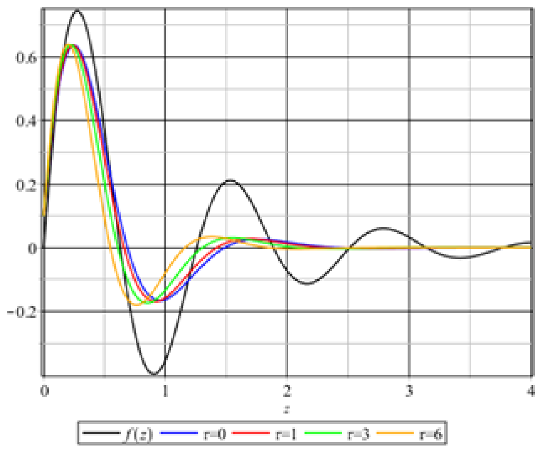

Fig. 2. The numerical convergence of the sequence $K_{\eta}^{r,\alpha}$ to the function $f(z)$, with $\eta = 25$ when $\alpha = 0.6$ and $r = 0, 1, 3, 6$.

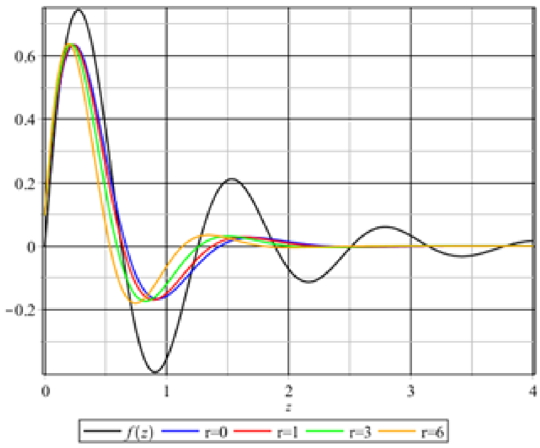

Fig. 3. The numerical convergence of the sequence $K_{\eta}^{r,\alpha}$ to the function $f(z)$, with $\eta = 25$ when $\alpha = 1$ and $r = 0, 1, 3, 6$.

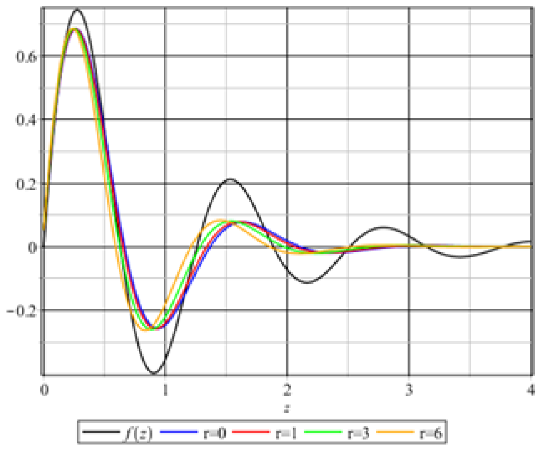

Fig. 4. The numerical convergence of the sequence $K_{\eta}^{r,\alpha}$ to the function $f(z)$, with $\eta = 50$ when $\alpha = 0.2$ and $r = 0, 1, 3, 6$.

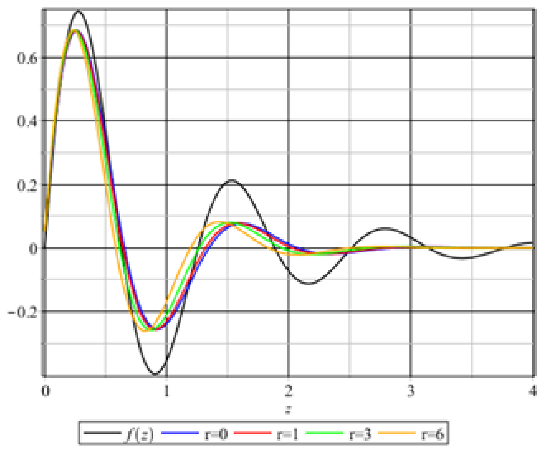

Fig. 5. The numerical convergence of the sequence $K_{\eta}^{r,\alpha}$ to the function $f(z)$, with $\eta = 50$ when $\alpha = 0.6$ and $r = 0, 1, 3, 6$.

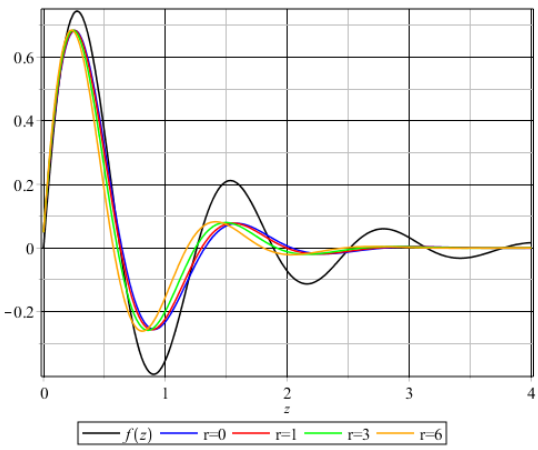

Fig. 6. The numerical convergence of the sequence $K_{\eta}^{r,\alpha}$ to the function $f(z)$, with $\eta = 50$ when $\alpha = 1$ and $r = 0, 1, 3, 6$.

Table 1. The average absolute errors $\sum_{j=0}^{v}\frac{|K_{\eta}^{r,\alpha}(f;z)-f(z)|}{v+1}$ for Example 5.1 with $h = 0.1$.

	$\boldsymbol{\eta = 25}$			$\boldsymbol{\eta = 50}$		
r	$\alpha = 0.2$	$\alpha = 0.6$	$\alpha = 1$	$\alpha = 0.2$	$\alpha = 0.6$	$\alpha = 1$
0	0.0800730	0.0782174	0.0764983	0.0569613	0.0555030	0.0540446
1	0.0715175	0.0620011	0.0533858	0.0615543	0.0531197	0.0459250
2	0.0946964	0.0796830	0.0662075	0.1418660	0.1288297	0.1177751
3	0.1490917	0.1270492	0.1093099	0.2976802	0.2780578	0.2623235
4	0.2385626	0.2068737	0.1785625	0.5872345	0.5448396	0.5168759
5	0.3761834	0.3242988	0.2897664	1.0950576	1.0226159	0.9626328
6	0.5710918	0.5011158	0.4420772	2.0219489	1.8532611	1.7470131

Table 1, discuss numerical results for different values of $r = 0,1,2,3,4,5,6$ and $\alpha = \{0.2, 0.6, 1\}$ with $\eta = 25, 50.$ From the above table, we observe the following:
The better approximation occurs at $\alpha = 1$ and $r = 1$. However, for $r = 0,2,3,4,5,6,$ the better approximation occurs at $\alpha = 1$.

6. Conclusion

This study contributes to the field of approximation theory by introducing a modified sequence of linear positive operators that generalizes and improves upon the classical Baskakov-Kantorovich and α-Baskakov-Kantorovich operators. The incorporation nonnegative parameters into their construction, endows the operators with greater flexibility and enhanced approximation capabilities. We first establish a convergence theorem for the proposed sequence of operators by employing a Korovkin-type result for sequences of linear positive operators on the interval $[0, \infty)$. Next, we define and derive a recurrence relation for the m-th order moment, which plays a crucial role in analyzing the ordinary approximation behavior of the sequence. Furthermore, we prove a Voronovskaja-type asymptotic formula for the operators. Numerical results demonstrate that the minimal approximation achieved when $\alpha = 1$, regardless of the value of r (tested for $r = 0,1,2,3,4,5,6$). whereas at $\alpha = 0.2,0.3,1$ the minimal error is attained when $r = 1$. While changes in α yield marginal improvements in accuracy, they significantly enhance the flexibility of the operators for adapting to diverse function classes. This adaptability makes the proposed operators particularly suitable for applications requiring fine-tuned control over convergence.

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تعميم لمتتابعة باسكاكوف-كانتوروفيتج معتمدة على معلَمتَين غير سالبتين

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معلومات البحث

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المراجعة	22 تشرين ثاني 2025
القبول	04 كانون أول 2025
النشر	31 كانون أول 2025

الكلمات المفتاحية

نظرية التقريب الاعتيادي، مؤثرات باسكاكوف، مؤثرات α باسكاكوف–كانتوروفيتج، الصيغة التقاربية لمبرهنة فرونفيسكيا.

الملخص

يقدّم هذا البحث متتابعة عامة من المؤثّرات الخطية الموجبة، يُشار إليها بمتتابعة باسكاكوف–كانتوروفيتج من النوع (r,α) ، والتي تعد اعمام لمتتابعة باسكاكوف–كانتوروفيتج الكلاسيكية من خلال إستخدام معلَمتين غير سالبتين r و α . تضفي هاتان المعلَمتان مرونةً للمتتابعة الناتجة، مما يتيح تحكما أدق في انسيابية التقريب. وقد تم إثبات تقارب هذه المتتابعة إلى أي دالة مستمرة ومقيدة التكامل على مجموعة الاعداد الحقيقية الموجبة باستخدام مبرهنة كوروفكين. علاوةً على ذلك، تم اشتقاق الصيغة المكافئة لمبرهنة فرونفيسكيا، والتي أعطت رؤيةً واضحةً لرتبة التقارب عندما تقترب η من اللانهاية. ولدعم النتائج النظرية، قدّم مثالًا عدديًّا طبّقت فيه هذه المتتابعة لتقريب دالة اختبارية. وأظهرت النتائج العددية أن المؤثرات المقترحة لا تحسن من دقة التقريب فقط بل انها تكشف عن مرونة في التعامل مع أنواع مختلفة من دوال الاختبار التي تحتوي على معلمات، ما يبرز إمكانات هذه المتتابعة المقترحة كأداة أكثر مرونة وفعالية في التقريب. ويسهم هذا البحث في تطوير تعميمات ذات معلمات للمؤثّرات الكلاسيكية تتميّز بخصائص تقريبية محسّنة.

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