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On the Role of the Projection Operator in the Chordal Transform Operator

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ABSTRACT

This paper investigates the projection operator in chordal transform connected with bounded linear operator on a separable Hilbert space. One projection operator is introduced by expanding a normalized form of the generalized derivation induced by the given operator, leading to a new formulation of a chordal transform. In addition, the Hilbert-Schmidt norm, Schatten P-norm and structural properties of the projection operator in chordal transform are examined in the case of self-adjoined operator, with providing the comparison between the new form of chordal transform and corresponding the generalized derivation.

1. Introduction

From [1], let ρ be an operator of H, when H is Hilbert space then ρ is called a projection if the following conditions are hold:
$\rho^{2} = \rho$ *and* $\rho^{*} = \rho$, when ρ^{*} be a self-adjoint of ρ.
If ρ is a projection operator on H, then it is linear and bounded.
Let B(H) represent the C^{*}-algebra of all bounded linear operators on a separable Hilbert space, as defined in [2].
The generalized derivation P_A of the projection operator Q on B(H) is defined as follows:

$$\rho_A = QXQ^{*} - X$$

for all of $X \in B(H)$.
If ρ is a projection and normed operator on $B(H)$, then the derivation ρ_A is said to be a generalized normal derivation [2]. Moreover, for $Q \in B(H)$ and all $X \in B(H)$, the formula of the chordal transformation G_Q as an operator on $B(H)$ is defined by

$$G_Q(X) = (|Q^{*}|^2 + I)^{-1/2}\, \rho_Q(X)\, (|Q|^2 + I)^{-1/2},$$

where Q is a projection operator on $B(H)$.

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The chordal transform has some geometric properties that resembles those of the chordal distance. Recall that the chordal distance between any two complex numbers a and b is given by

$$d(x,y)=\frac{|x-y|}{\sqrt{|x|^2+1}\sqrt{|y|^2+1}}.$$

It is obvious to see

$$d(x,y)\le 1$$

for all $x,y\in\mathbb{C}$, for more information, see [3, pp. 316-317].
Several authors ([4], [5], [6], [7], [8] and [9]) have been extensively studied the orthogonality of the range and the kernel with certain derivation.
Furthermore, for more transforms of operators in Hilbert Space, see [10],[11],[12], and other space [13].

2. Basic Concepts

This section covered some definitions and theorems that will be useful.

Definition (2.1) [12]
A bounded linear operator $Q\in B(H)$ on a separable Hilbert space H is called a compact operator if its singular values $S_1(Q)\ge S_2(Q)\ge\cdots\ge 0$ tends to zero,
i.e. the eigenvalues of $|\,Q\,|=(Q^*Q)^{1/2}$.

Definition (2.2) [2]
Unitary invariant norms of Spectral norm on $B(H)$ denoted by $|||.|||$ on a norm ideal $C_{|||.|||}$ associated with it, when $C_{|||.|||}$ is Banach space under the norm by satisfying the invariance property

$$|||P_1QP_2|||=|||Q|||$$

for all $Q\in C_{|||.|||}$.
Furthermore, for all unitary operator $P_1,P_2\in B(H)$, with applying the symmetry condition

$$|||P_3QP_4|||\le |||P_3|||\,\|P_4\|$$

for all $Q\in C_{|||.|||}$ and $P_3,P_4\in B(H)$[2],

Remark (2.3) [2]
Every unitary invariant norm of an operator is symmetric function of its singular value.

Definition (2.4) [14]
A compact operator $Q\in B(H)$ is called a trace class if the sum of its singular values is finite,

$$\|\,Q\,\|_1=\sum_{j=1}^{\infty}S_j\,(Q)<\infty.$$

The trace of Q is defined as

$$\mathrm{tr}\,(Q)=\sum_j\langle Q\,e_j,e_j\rangle$$

for any orthonormal basis (e_j) of H from [14].

Definition (2.5) [2]
Let $Q_1,Q_2\in B(H)$, the Hilbert- Schmidt become Hilbert space by the following condition

$$\langle Q_1,Q_2\rangle=trQ_2{}^*Q_1=trQ_1Q_2{}^*,$$

where tr is the trace functional.
Furthermore, for $\{e_i\}$ and $\{G_j\}$ are any orthonormal bases for H then the norm of Hilbert-Schmidt defined by

$$\|Q\|_2 = (tr\ Q^*Q\ )^{1/2} = \left(\sum_{i,j=1}^{\infty}\left|\langle QG_j, e_i\rangle\right|^2\right)^{1/2}.$$

Properties of the Trace Functional (2.6) [14]

Let $Q_1, Q_2 \in B(H)$ be trace class operators. The trace functional satisfies the following properties:

1. $\mathrm{tr}(\alpha Q_1 + \beta Q_2) = \alpha\,\mathrm{tr}(Q_1) + \beta\,\mathrm{tr}(Q_2)$, for all $\alpha, \beta \in \mathbb{C}$;

2. $\mathrm{tr}(Q_1 Q_2) = \mathrm{tr}(Q_2 Q_1)$;

3. $\mathrm{tr}\,({Q_1}^*) = \overline{\mathrm{tr}\,(Q_1)}$;

4. If $Q_1 \geq 0$, then $\mathrm{tr}\,(Q_1) \geq 0$;

5. $|\ \mathrm{tr}\,(Q_1)\ | \leq \|\ Q_1\ \|_1$, where $\|\ Q_1\ \|_1$ is the trace norm of Q_1;

6. $\mathrm{tr}(P^*QP) = \mathrm{tr}(Q)$, for any unitary operator $P \in B(H)$.

Definition (2.7) [2]
Let Q be a compact operator of $B(H)$, Schatten p −norms defined as

$$\|Q\|_p = \left(\sum_{j=1}^{\infty} S_j^p(Q)\right)^{1/p} \leq \infty,$$

where $S_1(Q) \geq S_2(Q) \geq \cdots \geq 0$ be a singular value of Q or the eigenvalues of $|Q| = (Q^*Q)^{1/2}$. The Schatten p-classes or norm C_p classified by the value of p, C_1 is called a trace class, C_2 is a Hilbert-Schmidt class, and C_∞-compact operator class when $p = 1$, $p = 2,$ and $p = \infty$ respectively.
Furthermore, for more inequalities for the Hilbert-Schmidt norm see [15].

3. Main Results

Lemma (3.1)
Let Q be a projection operator on $B(H)$ then $QXQ^* - X = QXQ - X$, for all $X \in B(H)$.
Proof.
Since q is projection operator, so $Q = Q^*$.
Thus, $QXQ^* - X = QXQ - X$ ■

Example (3.2)
Let $\acute{A} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, where A be a projection operator on $B(H)$ and $\acute{X} = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}$.
So $AXA^* - X = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & XA \end{pmatrix}$.
Then, $G_A(X) = (|A^*|^2 + I)^{-1/2}(AXA^* - X)(|A|^2 + I)^{-1/2}$

$$= \left(\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)^{-1}\begin{pmatrix} 0 & 0 \\ 0 & XA \end{pmatrix}$$

$$= \left(\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & XA \end{pmatrix}\right) = \left(\begin{pmatrix} 1 & -A \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & XA \end{pmatrix}\right).$$

Therefore, $G_A(X) = \begin{pmatrix} 0 & -XA \\ 0 & XA \end{pmatrix}$.

Theorem (3.3)
Let Q be a projection operator on $B(H)$ then $\left\|G_Q(X)\right\| \leq \left\|\rho_Q(X)\right\|$, for all $X \in B(H)$.

Proof.
Since Q is a projection operator on $B(H)$, so $Q^2 = Q^* = Q$.
And then w$|Q^*|^2 = Q^*Q = QQ = Q^2 = Q$, also $Q \geq 0$ so $Q \leq I$.
Now, $I \leq I + Q \leq 2I$

$$\frac{1}{2} \leq (I+Q)^{-1} \leq I$$

So $(I+Q)^{-1} \leq I \cdots (*)$

$$\begin{aligned} \text{Then, } \|G_Q(X)\| &= \left\|(|Q^*|^2 + I)^{-1/2}(QXQ^* - X)(|Q|^2 + I)^{-1/2}\right\| \\ &= \left\|(Q+I)^{-1/2}(QXQ - X)(Q+I)^{-1/2}\right\| \\ &= \|(Q+I)^{-1}(QXQ - X)\| \\ &= \|(Q+I)^{-1}\|\|(QXQ - X)\| \leq \|(QXQ - X)\| \cdots \{by(*)\}. \end{aligned}$$

Thus, $\|G_Q(X)\| \leq \|(QXQ - X)\|$
Hence, $\|G_Q(X)\| \leq \|\rho_Q(X)\|$ ■

Corollary (3.4)
Let Q be a projection operator on $B(H)$ of chordal transform G_Q then

$$\|G_Q(X)\| \leq \|X\|.$$

Proof.
From (3.2) we have:
$\|G_Q(X)\| \leq \|(QXQ - X)\|$,

$$\begin{aligned} \text{and } \|(QXQ - X)\| &\leq \|QXQ\| - \|X\| \\ &\leq \|QXQ\| + \|X\| \\ &= \|Q\|\|X\|\|Q\| + \|X\| \\ &\leq \|X\| + \|X\| = 2\|X\| \end{aligned}$$

Hence, $\|G_Q(X)\| \leq \|X\|$ ■

Corollary (3.5)
Let Q be a projection operator on $B(H)$ then $G_Q(X) = 0$ if and only if $QXQ = X$, for all $X \in B(H)$.
Proof.
Since Q is a projection operator on $B(H)$, so $Q^2 = Q^* = Q$.
Now, let assume that $G_Q(X) = 0$, we have to show $QXQ = X$.
$G_Q(X) = (Q+I)^{-1}(QXQ - X) = 0$.
Then $(QXQ - X) = 0$, since $(Q+I)^{-1} \neq 0$.
Thus, $QXQ = X$.
Conversely, let $QXQ = X$,
$so\ QXQ - X = 0$.
Then we have to prove $G_Q(X) = 0$
$G_Q(X) = (Q+I)^{-1}(QXQ - X) = 0$.
Hence, the proof is complete. ■

Remark (3.6)
From above corollary we can define $\ker G_Q$ as

$$\ker G_Q = \{X \in B(H) \backslash QXQ = X\}.$$

Corollary (3.7)
If Q is a projection operator on $B(H)$, then

$$|||G_Q(X)||| \leq |||\rho_Q(X)|||.$$

Proof.
From (2.2) and $(I+Q)^{-1} \leq I$, it follows:

$$\begin{aligned} |||G_Q(X)||| &= |||(Q+I)^{-1}(QXQ - X)||| \leq |||(Q+I)^{-1}|||\ |||QXQ - X||| \leq |||QXQ - X||| \\ &= |||\rho_Q(X)||| \end{aligned}$$

Thus, $|||G_Q(X)||| \leq |||\rho_Q(X)|||$ ■

Theorem (3.8)
If $Q, W \in B(H)$, when Q is a projection operator and normal, $W \in C_2$ with $QW^*Q = W^*$, then
$$\|\rho_Q\,(X) + W\|_2^2 = \|\rho_Q\,(X)\|_2^2 + \|W\|_2^2,$$
for all $X \in B(H)$.
Proof.
From (2.5) and (2.7) we can have
$\left\|\rho_Q\,(X) + W\right\|_2^2 = \langle \rho_Q\,(X) + W, \rho_Q\,(X) + W\rangle = \|\rho_Q\,(X)\|_2^2 + 2\,Re\langle \rho_Q\,(X), W\rangle + \|W\|_2^2$
Now we have to show $\langle \rho_Q\,(X), W\rangle = 0$.
Since $QW^*Q = W^*$, and Q is projection operator,
so $QW^*\,Q^* = W^*$ and $QW^*Q = Q^*$, then we have
$\langle \rho_Q\,(X), W\rangle = tr\left(W^*\rho_Q\,(X)\right) = tr(W^*QXQ - QW^*QX)$
$= tr(W^*QXQ) - tr(QW^*QX)$
$= tr(QW^*QX) - tr(QW^*QX)$
$\mathrm{tr}\big((QW^*Q - QW^*Q)X\big) = tr(0) = 0.$
Thus, $\langle \rho_Q\,(X), W\rangle = 0$
Hence, $\left\|\rho_Q\,(X) + W\right\|_2^2 = \left\|\rho_Q\,(X)\right\|_2^2 + \|W\|_2^2$ ■

Theorem (3.9)
Let $Q, W \in B(H)$, where Q is a projection operator and normal, $W \in C_2$ and $QW^*\mathrm{Q} = W^*$, then
$$\left\|G_Q\,(X) + W\right\|_2^2 = \left\|G_Q\,(X)\right\|_2^2 + \|W\|_2^2.$$
Proof.
Since $Q \in C_2$, then let $G_Q(X) + W \notin C_2$, so $G_Q(X) \notin C_2$.
So, by (2.5) and (2.7) $\left\|G_Q\,(X)\right\|_2^2 = \infty$.
Then $\left\|G_Q\,(X)\right\|_2^2 + \|W\|_2^2 = \infty$.
Now suppose that $G_Q(X) + Q \in C_2$.
Since Q is projection operator and $QW^*Q = W^*$.
Then $Q^*W^*Q = W^*$or $QW^*Q^* = W^*$.
Thus, $W^*G_Q(X) \in C_2$.

$$\begin{aligned}
\mathrm{tr}\,W^*G_Q(X) &= \mathrm{tr}\,W^*\left((|Q^*|^2 + I)^{-1/2}(QXQ^* - X)(|Q|^2 + I)^{-1/2}\right)\\
&= \mathrm{tr}\,W^*(|Q^*|^2 + I)^{-1/2}QXQ^*(|Q|^2 + I)^{-1/2} - \mathrm{tr}\,W^*(|Q^*|^2 + I)^{-1/2}X(|Q|^2 + I)^{-1/2}\\
&= \mathrm{tr}\,Q^*W^*Q(|Q^*|^2 + I)^{-1/2}X(|Q|^2 + I)^{-1/2} - \mathrm{tr}\,W^*(|Q^*|^2 + I)^{-1/2}X(|Q|^2 + I)^{-1/2}\\
&= \mathrm{tr}\,W^*(|Q^*|^2 + I)^{-1/2}X(|Q|^2 + I)^{-1/2} - \mathrm{tr}\,W^*(|Q^*|^2 + I)^{-1/2}X(|Q|^2 + I)^{-1/2}\\
&= 0
\end{aligned}$$

Therefore, $\mathrm{tr}\,W^*G_Q(X) = 0$.

$$\begin{aligned}
\text{Hence, } \left\|G_Q\,(X) + W\right\|_2^2 &= \left\|G_Q\,(X)\right\|_2^2 + 2Re\;tr\;W^*G_Q(X) + \|W\|_2^2\\
&= \left\|G_Q\,(X)\right\|_2^2 + \|W\|_2^2 \blacksquare
\end{aligned}$$

4. Conclusion

This paper introduced new formula for generalized derivation $\rho_Q(X)$, where Q is defined as a projection operator on $B(H)$. This simplifies the Chordal transform formula for solving problems under particular projection conditions ($Q^2 = Q^* = Q$). Furthermore, utilizing the norm operator in some theory, norm equality and inequality between $G_Q(X)$ and $\rho_Q(X)$ are shown in a straightforward and understandable manner. Finally, in certain results, we illustrated the Kernel of Chordal Transform, which may lead to further results in future work.

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حول المؤثر الأسقاطي في مؤثر التحويل الوتري

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الكلمات المفتاحية

المؤثر الأسقاطي، مؤثر التحويل الوتري، معيار شاتن-P ، معيار هلبرت-شميدت.

الملخص

تتناول هذه الورقة البحثية مُؤثر الإسقاط في التحويل الوتري المرتبط بمؤثر خطي محدود على فضاء هيلبرت قابل للفصل. تم تقديم مُؤثر إسقاطي جديد من خلال توسيع صيغة للاشتقاق المُعمَّم الناتج عن هذا المُؤثر، مما يُؤدي إلى صياغة جديدة للتحويل الوتري. إضافةً إلى ذلك، تم اختبار خصائص مُؤثر الإسقاط في التحويل الوتري، بما في ذلك معيار هيلبرت-شميدت، ومعيار شاتن-P، والخصائص البنيوية، وذلك في حالة المُؤثر ذاتي الاقتران، مع تقديم مقارنة بين الصيغة الجديدة للتحويل الوتري والاشتقاق المُعمَّم المُقابل لها.

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