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A Review on Cartan's Structure Equations for Certain Classes of Almost Contact Metric Manifolds

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ABSTRACT

This article is review the first group of Cartan's structure equations for certain classes of almost contact metric manifolds. These classes are divide into two collections, the first involve the irreducible classes such as cosymplectic class, Kenmotsu class, Sasakian class, C_9- class, C_{11}- class, and C_{12}- class. The second include normal class of Killing type (CNK-class), nearly Kenmotsu class, NC_{10}- class, NC_{11}- class, nearly cosymplectic class, and Kenmotsu type class.

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1. Introduction

Cartan's structure equations play an important role in Riemannian geometry. According to this important role many researchers determined the first group of Cartan's structure equations (1^{st}-CSE). So, some of them can mentioned as Kirichenko [1] derived the 1^{st}-CSE of almost contact metric manifolds that classified by Chinea and Gonzalez [2]. Volkova [3] given the 1^{st}-CSE of normal class of Killing type (briefly CNK-class). Umnova [4] deduced the 1^{st}-CSE of Kenmotsu manifolds and nearly Kenmotsu (generalized Kenmotsu) manifolds. Dondukova [5] established the 1^{st}-CSE of cosymplectic manifolds and Sasakian manifolds. The studying of the 1^{st}-CSE continued up to now by many authors, especially Rustanov et al. in [6, 7, 8].

2. Preliminaries

We denote by M^{2n+1} and g, the smooth manifold M of dimension $2n+1$ and the Riemannian metric respectively. Note that $X(M)$ is the module of whole vector fields on M.

Definition 2.1. [2] If a Riemannian manifold (M^{2n+1}, g) is provided by a triple of a structure tensor (ξ, η, Φ), where Φ is a (1, 1)-tensor over M, ξ is a vector field on M and η is a 1-form of M, such that $\forall\, U, V \in X(M)$, the following hold:

$$\Phi(\xi) = 0; \quad \eta(\xi) = 1; \quad \eta \circ \Phi = 0; \quad \Phi^2 + id = \eta \otimes \xi;$$
$$g(\Phi U, \Phi V) + \eta(U)\eta(V) = g(U, V),$$

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then it is called an almost contact metric (ACM-) manifold and denoted by $(M^{2n+1},\xi,\eta,\Phi,g)$.

There are several classes of ACM-manifolds $(M^{2n+1},\xi,\eta,\Phi,g)$. We refer to some of these classes as the following:

Classes	**Defining conditions**
Cosymplectic [9]	$\nabla_X(\Phi)Y=0$
Nearly cosymplectic [10]	$\nabla_X(\Phi)Y+\nabla_Y(\Phi)X=0$
Kenmotsu [11]	$\nabla_X(\Phi)Y=-g(X,\Phi Y)\xi-\eta(Y)\Phi X$
Sasakian [12]	$\nabla_X(\Phi)Y=g(X,Y)\xi-\eta(Y)X$
C_9 [8]	$\nabla_X(\Phi)Y=\eta(Y)\nabla_{\Phi X}\xi-g(\Phi X,\nabla_Y\xi)\xi$
C_{11} [6]	$\nabla_X(\Phi)Y=\eta(X)\Phi\circ\nabla_\xi(\Phi)\Phi Y$
C_{12} [13]	$\nabla_X(\Phi)Y=-\eta(X)\{\eta(Y)\Phi(\nabla_\xi\xi)+g(\nabla_\xi\xi,\Phi Y)\xi\}$
CNK [3]	Normal and $\nabla_X(\eta)Y+\nabla_Y(\eta)X=0$
Nearly Kenmotsu [14]	$\nabla_X(\Phi)Y+\nabla_Y(\Phi)X=-\eta(Y)\Phi X-\eta(X)\Phi Y$
NC_{10} [15]	$\nabla_X(\Phi)Y+\nabla_Y(\Phi)X = \xi\nabla_X(\eta)\Phi Y+\xi\nabla_Y(\eta)\Phi X+\eta(X)\nabla_{\Phi Y}\xi+\eta(Y)\nabla_{\Phi X}\xi$
NC_{11} [7]	$\nabla_X(\Phi)Y+\nabla_Y(\Phi)X=\eta(X)\Phi\circ\nabla_\xi(\Phi)\Phi Y+\eta(Y)\Phi\circ\nabla_\xi(\Phi)\Phi X$
Kenmotsu type [16]	$\nabla_X(\Phi)Y-\nabla_{\Phi X}(\Phi)\Phi Y=-\eta(Y)\Phi X$

for all $X,Y\in X(M)$, where ∇ is the Levi-Civita connection (Riemannian connection). Moreover, an ACM-manifold $(M^{2n+1},\xi,\eta,\Phi,g)$ is called normal if $2N+\xi\otimes d\eta=0$, where

$$N(X,Y)=\frac{1}{4}\Big([\Phi X,\Phi Y]+\Phi^2[X,Y]-\Phi[\Phi X,Y]-\Phi[X,\Phi Y]\Big),$$

is the Nijenhuis tensor of the structure tensor Φ (see [3]).

Theorem 2.2. [17] (Cartan's structure equations) Suppose that (M^n,g) is Riemannian manifold, θ is the connection form of Riemannian connection ∇, R is Riemannian curvature tensor of type (3, 1) and $\{\omega^1,\dots,\omega^n\}$ is the dual frame to the basis frame $\{E_1,\dots,E_n\}$ of $X(M)$. Then the following hold:

(1) $\boldsymbol{d\omega^i=-\theta^i_j\wedge\omega^j}$; (first group)

(2) $\boldsymbol{d\theta^i_j=-\theta^i_k\wedge\theta^k_j+\frac{1}{2}R^i_{jkl}\,\omega^k\wedge\omega^l}$, (second group)

where θ^i_j and R^i_{jkl} are the components of θ and R respectively, whereas, $i,j,k,l=1,\dots,n$.

On the other hand, Kirichenko [1] introduced a new method called associated G-structure space, such that the tensors g and Φ of ACM-manifold M^{2n+1} are given in the following formulae [18]:

$$(g_{kl})=\begin{pmatrix}1 & o & o\\ o & O & I_n\\ o & I_n & O\end{pmatrix};\qquad (\Phi^k_l)=\begin{pmatrix}o & o & o\\ o & \sqrt{-1}I_n & O\\ o & O & -\sqrt{-1}I_n\end{pmatrix}$$

where $k,l=0,1,\dots,2n$ and I_n is $n\times n$ identity matrix.

Moreover, on the associated G-structure space, Kirichenko [1] constructed the first group of Cartan's structure equations of Theorem 2.2 in the forms of theorem below.

Theorem 2.3. [19] Suppose that $(M^{2n+1},\xi,\eta,\Phi,g)$ is an ACM-manifold. Then the first group of Cartan's structure equations on associated G-structure space is given by:

(1) $d\omega^a=-\theta^a_b\wedge\omega^b+B^{ab}{}_c\,\omega^c\wedge\omega_b+B^{abc}\,\omega_b\wedge\omega_c+B^a_b\,\omega\wedge\omega^b+B^{ab}\,\omega\wedge\omega_b$;
(2) $d\omega_a=\theta^b_a\wedge\omega_b+B_{ab}{}^c\,\omega_c\wedge\omega^b+B_{abc}\,\omega^b\wedge\omega^c+B_a{}^b\,\omega\wedge\omega_b+B_{ab}\,\omega\wedge\omega^b$;
(3) $d\omega=C_{bc}\,\omega^b\wedge\omega^c+C^{bc}\,\omega_b\wedge\omega_c+C^b_c\,\omega^c\wedge\omega_b+C_b\,\omega\wedge\omega^b+C^b\,\omega\wedge\omega_b$,
where $C^b_c=B^b{}_c-B_c{}^b$.

3. Structure Equations of Chinea and Gonzalez Classes

This section specified to determined Cartan's structure equations of certain irreducible classes of ACM-manifolds that mentioned in Chinea and Gonzalez classification [2]. Precisely, this section devoted to determined Cartan's structure equations of cosymplectic class, Kenmotsu class, Sasakian class, C_9 −class, C_{11} −class, and C_{12} −class respectively as follow:

Theorem 3.1. [5] The first group of structure equations of Cartan for cosymplectic manifolds are given by:

1. $d\omega^a=-\theta^a_b\wedge\omega^b$;
2. $d\omega_a=\theta^b_a\wedge\omega_b$;
3. $d\omega=0$.

Theorem 3.2. [5] The first group of structure equations of Cartan for Kenmotsu manifolds are given by:

1. $d\omega^a=-\theta^a_b\wedge\omega^b+\omega\wedge\omega^a$;
2. $d\omega_a=\theta^b_a\wedge\omega_b+\omega\wedge\omega_a$;
3. $d\omega=0$.

Theorem 3.3. [5] The first group of structure equations of Cartan for Sasakian manifolds are given by:

1. $d\omega^a=-\theta^a_b\wedge\omega^b-\sqrt{-1}\,\omega\wedge\omega^a$;
2. $d\omega_a=\theta^b_a\wedge\omega_b+\sqrt{-1}\,\omega\wedge\omega_a$;
3. $d\omega=-2\sqrt{-1}\,\omega^a\wedge\omega_a$.

Theorem 3.4. [8] The first group of structure equations of Cartan for C_9-manifolds are given by:

1. $d\omega^a=-\theta^a_b\wedge\omega^b+F^{ab}\,\omega_b\wedge\omega$;
2. $d\omega_a=\theta^b_a\wedge\omega_b+F_{ab}\,\omega^b\wedge\omega$;
3. $d\omega=0$,

where $F^{ab}=F^{ba}$; $F_{ab}=F_{ba}$; $F_{ab}=\overline{F^{ab}}$.

Theorem 3.5. [6] The first group of structure equations of Cartan for C_{11}-manifolds are given by:

1. $d\omega^a=-\theta^a_b\wedge\omega^b+B^{ab}\,\omega\wedge\omega_b$;
2. $d\omega_a=\theta^b_a\wedge\omega_b+B_{ab}\,\omega\wedge\omega^b$;

3. $d\omega = 0,$

where $B^{ab} = -B^{ba}; \quad B_{ab} = -B_{ba}; \quad B_{ab} = \overline{B^{ab}}.$

Theorem 3.6. [13] The first group of structure equations of Cartan for C_{12}-manifolds are given by:

1. $d\omega^a = -\theta^a_b \wedge \omega^b;$
2. $d\omega_a = \theta^b_a \wedge \omega_b;$
3. $d\omega = C_b\ \omega \wedge \omega^b + C^b\ \omega \wedge \omega_b.$

4. Cartan's Structure Equations of Not Irreducible Classes

This section specified to determined Cartan's structure equations of certain classes of ACM-manifolds which are not irreducible. Precisely, this section devoted to determined Cartan's structure equations of CNK-class, nearly Kenmotsu class, NC_{10}-class, NC_{11}-class, nearly cosymplectic class, and Kenmotsu type class respectively as follow:

Theorem 4.1. [3] The first group of structure equations of Cartan for CNK-class on the space of associated G-structure are given by:

1. $d\omega^a = -\theta^a_b \wedge \omega^b + B^{ab}_{\ \ c}\ \omega^c \wedge \omega_b - B^a_{\ b}\ \omega^b \wedge \omega;$
2. $d\omega_a = \theta^b_a \wedge \omega_b + B_{ab}^{\ \ c}\ \omega_c \wedge \omega^b - B_a^{\ b}\ \omega_b \wedge \omega;$
3. $d\omega = 2B^{\ b}_{a}\ \omega^a \wedge \omega_b,$

where $B^a_{\ b} = -B_b^{\ a}.$

Theorem 4.2. [14] The first group of structure equations of Cartan for nearly Kenmotsu manifolds on the space of associated G-structure are given by:

1. $d\omega^a = -\theta^a_b \wedge \omega^b + C^{abc}\ \omega_b \wedge \omega_c + \frac{3}{2}F^{ab}\ \omega_b \wedge \omega - \omega^a \wedge \omega;$
2. $d\omega_a = \theta^b_a \wedge \omega_b + C_{abc}\ \omega^b \wedge \omega^c + \frac{3}{2}F_{ab}\ \omega^b \wedge \omega - \omega_a \wedge \omega;$
3. $d\omega = F_{ab}\ \omega^a \wedge \omega^b + F^{ab}\ \omega_a \wedge \omega_b,$

where $F^{ab} + F^{ba} = 0; \quad F_{ab} + F_{ba} = 0; \quad C^{[abc]} = C^{abc}; \quad C_{[abc]} = C_{abc}; \quad \overline{F^{ab}} = F_{ab}; \quad \overline{C^{abc}} = C_{abc}.$

Theorem 4.3. [15] The first group of structure equations of Cartan for NC_{10}- manifolds on the space of associated G-structure are given by:

1. $d\omega^a = -\theta^a_b \wedge \omega^b + C^{abc}\ \omega_b \wedge \omega_c + F^{ab}\ \omega_b \wedge \omega;$
2. $d\omega_a = \theta^b_a \wedge \omega_b + C_{abc}\ \omega^b \wedge \omega^c + F_{ab}\ \omega^b \wedge \omega;$
3. $d\omega = F_{ab}\ \omega^a \wedge \omega^b + F^{ab}\ \omega_a \wedge \omega_b,$

where $F^{ab} + F^{ba} = 0; \quad F_{ab} + F_{ba} = 0; \quad C^{[abc]} = C^{abc}; \quad C_{[abc]} = C_{abc}; \quad \overline{F^{ab}} = F_{ab}; \quad \overline{C^{abc}} = C_{abc}.$

Theorem 4.4. [7] The first group of structure equations of Cartan for NC_{11}- manifolds on the space of associated G-structure are given by:

1. $d\omega^a = -\theta^a_b \wedge \omega^b + B^{abc}\ \omega_b \wedge \omega_c + B^{ab}\ \omega \wedge \omega_b;$

2. $d\omega_a = \theta_a^b \wedge \omega_b + B_{abc}\, \omega^b \wedge \omega^c + B_{ab}\, \omega \wedge \omega^b;$

3. $d\omega = 0,$

where $B^{ab} + B^{ba} = 0;\quad B_{ab} + B_{ba} = 0;\quad B^{[abc]} = B^{abc};\quad B_{[abc]} = B_{abc};\quad \overline{B^{ab}} = B_{ab};\quad \overline{B^{abc}} = B_{abc}.$

Theorem 4.5. [10] The first group of structure equations of Cartan for nearly cosymplectic manifolds on the space of associated G-structure are given by:

1. $d\omega^a = -\theta_b^a \wedge \omega^b + C^{abc}\, \omega_b \wedge \omega_c + \frac{3}{2} F^{ab}\, \omega_b \wedge \omega;$

2. $d\omega_a = \theta_a^b \wedge \omega_b + C_{abc}\, \omega^b \wedge \omega^c + \frac{3}{2} F_{ab}\, \omega^b \wedge \omega;$

3. $d\omega = F_{ab}\, \omega^a \wedge \omega^b + F^{ab}\, \omega_a \wedge \omega_b,$

where $F^{ab} + F^{ba} = 0;\quad F_{ab} + F_{ba} = 0;\quad C^{[abc]} = C^{abc};\quad C_{[abc]} = C_{abc};\quad \overline{F^{ab}} = F_{ab};\quad \overline{C^{abc}} = C_{abc}.$

Theorem 4.6. [16] The manifold of Kenmotsu type has the following Cartan's structure equations (first group):

1. $d\omega^a = -\theta_b^a \wedge \omega^b + B^{ab}{}_c\, \omega^c \wedge \omega_b - \omega^a \wedge \omega;$
2. $d\omega_a = \theta_a^b \wedge \omega_b + B_{ab}{}^c\, \omega_c \wedge \omega^b - \omega_a \wedge \omega;$
3. $d\omega = 0.$

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مراجعة حول معادلات كارتان التركيبية لبعض فئات المنطويات المترية التلامسية التقريبية

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الكلمات المفتاحية

منطويات كينموتسو، معادلات كارتان التركيبية، منطويات ساساكي، منطويات كوسيمبلكتك التقريبي.

الملخص

هذه المقالة هي استعراض للنتائج السابقة حول معادلات كارتان التركيبية لبعض فئات المنطويات المترية التلامسية التقريبية. هذه الفئات من المنطويات قمنا بتقسيمها الى عائلتين، بحيث ان العائلة الأولى تتضمن الفئات غير القابلة للتحليل مثل فئة كوسيمبلكتك وفئة كينموتسو وفئة ساساكي والفئة - C_9 والفئة - C_{11} والفئة - C_{12}. اما العائلة الثانية فتشمل الفئة الطبيعية من النوع المعدوم (أي الفئة – CNK) و فئة كينموتسو التقريبي وفئة - NC_{10} وفئة - NC_{11} وفئة كوسيمبلكتك التقريبي والفئة من نوع كينموتسو.

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